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Physics and maths resources

OCR AS Level Further Mathematics A

H235/Y531 - Pure Core

Monday 15 May 2023

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the OCR Monday 15 May 2023 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

The roots of the equation $4x^4 - 2x^3 - 3x + 2 = 0$ are α, β, γ and δ. By using a suitable substitution, find a quartic equation whose roots are $\alpha+2$, $\beta+2$, $\gamma+2$ and $\delta+2$, giving your answer in the form $at^4 + bt^3 + ct^2 + dt + e = 0$, where a, b, c, d and e are integers.

Model solution

A root is increased by 2, so put $t = x + 2$, hence $x = t - 2$. The new variable therefore has exactly the required roots.

$$4(t-2)^4 - 2(t-2)^3 - 3(t-2) + 2 = 0.$$

$$(t-2)^4 = t^4 - 8t^3 + 24t^2 - 32t + 16, \qquad (t-2)^3 = t^3 - 6t^2 + 12t - 8.$$

$$\begin{aligned} 0 &= 4(t^4 - 8t^3 + 24t^2 - 32t + 16) \\ &\quad - 2(t^3 - 6t^2 + 12t - 8) - 3(t-2) + 2 \\ &= 4t^4 - 34t^3 + 108t^2 - 155t + 88. \end{aligned}$$

$4t^4 - 34t^3 + 108t^2 - 155t + 88 = 0.$ ***(5)***

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Question 2

Question 2

The lines L_1 and L_2 have the following equations:

$$L_1: \mathbf{r} = \begin{pmatrix} -5 \\ 6 \\ 15 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -2 \\ -2 \end{pmatrix}, \qquad L_2: \mathbf{r} = \begin{pmatrix} 24 \\ 1 \\ -5 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}.$$

(a) Show that L_1 and L_2 intersect, giving the position vector of the point of intersection.
(b) Find the equation of the line which intersects L_1 and L_2 and is perpendicular to both. Give your answer in cartesian form.

Model solution

(a) At an intersection the position vectors are equal, so equate corresponding components.

$$-5 + 5\lambda = 24 + 3\mu, \qquad 6 - 2\lambda = 1 + \mu, \qquad 15 - 2\lambda = -5 - 4\mu.$$

$$5\lambda - 3\mu = 29, \qquad 2\lambda + \mu = 5.$$

$$\mu = 5 - 2\lambda \implies 5\lambda - 3(5 - 2\lambda) = 29 \implies \lambda = 4, \quad \mu = -3.$$

The third component checks the result: $15 - 2(4) = 7 = -5 - 4(-3)$. *Thus the lines do intersect, at*

$$\mathbf{r} = \begin{pmatrix} -5 \\ 6 \\ 15 \end{pmatrix} + 4 \begin{pmatrix} 5 \\ -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 15 \\ -2 \\ 7 \end{pmatrix}.$$

The point of intersection has position vector $\begin{pmatrix} 15 \\ -2 \\ 7 \end{pmatrix}$. *(5)*

(b) A direction perpendicular to both lines is the cross product of their direction vectors.

$$\begin{pmatrix} 5 \\ -2 \\ -2 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 10 \\ 14 \\ 11 \end{pmatrix}.$$

Use the intersection point and this direction vector to write the cartesian equation.

$$\frac{x - 15}{10} = \frac{y + 2}{14} = \frac{z - 7}{11}. \qquad \textit{(3)}$$

Question 3

Question 3

In this question you must show detailed reasoning.
In this question the principal argument of a complex number lies in the interval $[0, 2\pi)$. Complex numbers z_1 and z_2 are defined by $z_1 = 3 + 4i$ and $z_2 = -5 + 12i$.
(a) Determine $z_1 z_2$, giving your answer in the form $a + bi$.
(b) Express z_2 in modulus-argument form.
(c) Verify, by direct calculation, that $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$.

Model solution

(a) Expand the product and use $i^2 = -1$.

$$\begin{aligned} z_1 z_2 &= (3 + 4i)(-5 + 12i) \\ &= -15 + 36i - 20i + 48i^2 \\ &= -63 + 16i. \end{aligned}$$

$z_1 z_2 = -63 + 16i.$ *(2)*

(b) The modulus is the distance from the origin. Since z_2 lies in quadrant II, its principal argument is π minus the reference angle.

$$|z_2| = \sqrt{(-5)^2 + 12^2} = 13, \qquad \arg(z_2) = \pi - \tan^{-1}\left(\frac{12}{5}\right) = 1.965587\ldots$$

$z_2 = 13\big(\cos 1.97 + i \sin 1.97\big)$, *with the angle in radians.* *(3)*

(c) Calculate the two sides, taking account of the quadrants.

$$\arg(z_1 z_2) = \pi - \tan^{-1}\left(\frac{16}{63}\right) = 2.892882\ldots$$

$$\begin{aligned} \arg(z_1) + \arg(z_2) &= \tan^{-1}\left(\frac{4}{3}\right) + 1.965587\ldots \\ &= 0.927295\ldots + 1.965587\ldots \\ &= 2.892882\ldots \end{aligned}$$

$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) = 2.892882\ldots$, *as required.* *(3)*

Question 4

Question 4

The vector $\mathbf{p}$, all of whose components are positive, is given by

$$\mathbf{p} = \begin{pmatrix} a^2 \\ a-5 \\ 26 \end{pmatrix}$$

where a is a constant. You are given that $\mathbf{p}$ is perpendicular to the vector $\begin{pmatrix} 2 \\ 6 \\ -3 \end{pmatrix}$. Determine the value of a.

Model solution

Perpendicular vectors have scalar product zero.

$$\begin{pmatrix} a^2 \\ a-5 \\ 26 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 6 \\ -3 \end{pmatrix} = 0$$

$$2a^2 + 6(a-5) - 3(26) = 0 \quad \Longrightarrow \quad a^2 + 3a - 54 = 0 \quad \Longrightarrow \quad (a-6)(a+9) = 0.$$

The candidate $a = -9$ gives $a - 5 = -14$, not a positive component, so it must be rejected.

$a = 6$. ***(4)***

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Question 5

Question 5

In this question you must show detailed reasoning.
The roots of the equation $5x^2 - 3x + 12 = 0$ are α and β. By considering the symmetric functions of the roots, $\alpha + \beta$ and $\alpha\beta$, determine the exact value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$.

Model solution

Vieta's formulae give the two symmetric functions directly from the quadratic.

$$\alpha + \beta = \frac{3}{5}, \qquad \alpha\beta = \frac{12}{5}.$$

Rewrite the required expression in terms of these quantities.

$$\begin{aligned}\frac{1}{\alpha^2} + \frac{1}{\beta^2} &= \frac{\alpha^2 + \beta^2}{\alpha^2\beta^2} \\ &= \frac{(\alpha+\beta)^2 - 2\alpha\beta}{(\alpha\beta)^2} \\ &= \frac{\left(\frac{3}{5}\right)^2 - 2\left(\frac{12}{5}\right)}{\left(\frac{12}{5}\right)^2} \\ &= -\frac{37}{48}.\end{aligned}$$

$-\frac{37}{48}$. ***(4)***

Question 6

Question 6

Prove by induction that $4 \times 8^n + 66$ is divisible by 14 for all integers $n \geq 0$.

Model solution

Induction is appropriate because the claim is made for every non-negative integer. For the base case $n = 0$,

$$4 \times 8^0 + 66 = 4 + 66 = 70 = 5 \times 14,$$

which is divisible by 14.
Assume the statement is true for $n = k$. *Thus, for some integer* p,

$$4 \times 8^k + 66 = 14p.$$

For the next case, rewrite the expression to contain the assumed expression.

$$\begin{aligned} 4 \times 8^{k+1} + 66 &= 32 \times 8^k + 66 \\ &= 8(4 \times 8^k + 66) - 462 \\ &= 8(14p) - 14(33) \\ &= 14(8p - 33). \end{aligned}$$

This is divisible by 14. *Hence truth for* $n = k$ *implies truth for* $n = k + 1$; *together with the base case, the result is true by induction.*

$4 \times 8^n + 66$ *is divisible by* 14 *for all integers* $n \geq 0$. ***(6)***

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Question 7

Question 7

In this question you must show detailed reasoning.
Matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} a & -6 & a-3 \\ a+9 & a & 4 \\ 0 & -13 & a-1 \end{pmatrix}$$

where a is a constant. Find all possible values of a for which $\det \mathbf{A}$ has the same value as it has when $a = 2$.

Model solution

Expand the determinant along the first row, then compare it with the value at $a = 2$.

$$\begin{aligned} \det \mathbf{A} &= a\big(a(a-1) - 4(-13)\big) - (-6)\big((a+9)(a-1) - 0\big) \\ &\quad + (a-3)\big((a+9)(-13) - 0\big) \\ &= a^3 - 8a^2 + 22a + 297. \end{aligned}$$

$$\det \mathbf{A}\big|_{a=2} = 2^3 - 8(2^2) + 22(2) + 297 = 317.$$

$$a^3 - 8a^2 + 22a + 297 = 317 \implies a^3 - 8a^2 + 22a - 20 = 0.$$

Since $a = 2$ *is a root, factor out* $a - 2$.

$$(a-2)(a^2 - 6a + 10) = 0.$$

$$a^2 - 6a + 10 = 0 \implies (a-3)^2 = -1 \implies a = 3 \pm i.$$

$a = 2,\ 3 + i$, *or* $3 - i$. *(6)*

Question 8

Question 8

(a) Solve the equation $\omega + 2 + 7i = 3\omega^* - i$.
(b) Prove algebraically that, for non-zero z, $z = -z^*$ if and only if z is purely imaginary.
(c) The complex numbers z and z^* are represented on an Argand diagram by the points A and B respectively.
(i) State, for any z, the single transformation which transforms A to B.
(ii) Use a geometric argument to prove that $z = z^*$ if and only if z is purely real.

Model solution

(a) Write $\omega = a + bi$, so $\omega^* = a - bi$, then equate real and imaginary parts.

$$(a + bi) + 2 + 7i = 3(a - bi) - i.$$

$$a + 2 = 3a, \qquad b + 7 = -3b - 1,$$

$$a = 1, \qquad b = -2.$$

$\omega = 1 - 2i$. *(4)*

(b) If z is purely imaginary, write $z = ki$, where k is real. Then

$$z^* = -ki = -z.$$

Conversely, let $z = r + si$, where r and s are real. If $z = -z^*$, then

$$r + si = -(r - si) = -r + si \implies 2r = 0 \implies r = 0.$$

Thus $z = si$, so z is purely imaginary.

For non-zero z, $z = -z^*$ if and only if z is purely imaginary. *(2)*

(c)(i) Conjugation preserves the real coordinate and reverses the imaginary coordinate.

Reflection in the real axis. *(1)*

(c)(ii) Conjugation is reflection in the real axis. If $z = z^*$, then A and B coincide, so A is invariant under this reflection. Such a point lies on the mirror line, the real axis, so z is purely real. Conversely, a point representing a purely real number lies on the real axis and is invariant under its reflection. Hence $A = B$, so $z = z^*$.

$z = z^*$ if and only if z is purely real. *(2)*

Question 9

Question 9

Matrix $\mathbf{R}$ is given by

$$\mathbf{R} = \begin{pmatrix} a & 0 & -b \\ 0 & 1 & 0 \\ b & 0 & a \end{pmatrix}$$

where a and b are constants.
(a) Find $\mathbf{R}^2$ in terms of a and b.

The constants a and b are given by $a = \frac{\sqrt{2}}{4}(\sqrt{3}+1)$ and $b = \frac{\sqrt{2}}{4}(\sqrt{3}-1)$.
(b) By determining exact expressions for ab and $a^2 - b^2$ and using the result from part (a), show that

$$\mathbf{R}^2 = k \begin{pmatrix} \sqrt{3} & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & \sqrt{3} \end{pmatrix}$$

where k is a real number whose value is to be determined.
(c) Find $\mathbf{R}^6$, $\mathbf{R}^{12}$ and $\mathbf{R}^{24}$.
(d) Describe fully the transformation represented by $\mathbf{R}$.

Model solution

(a) Multiply the matrix by itself, taking each row times each column.

$$\mathbf{R}^2 = \begin{pmatrix} a & 0 & -b \\ 0 & 1 & 0 \\ b & 0 & a \end{pmatrix} \begin{pmatrix} a & 0 & -b \\ 0 & 1 & 0 \\ b & 0 & a \end{pmatrix}$$
$$= \begin{pmatrix} a^2 - b^2 & 0 & -2ab \\ 0 & 1 & 0 \\ 2ab & 0 & a^2 - b^2 \end{pmatrix}.$$

$$\mathbf{R}^2 = \begin{pmatrix} a^2 - b^2 & 0 & -2ab \\ 0 & 1 & 0 \\ 2ab & 0 & a^2 - b^2 \end{pmatrix}. \quad \textit{(2)}$$

(b) Find the two combinations occurring in $\mathbf{R}^2$. For $a^2 - b^2$, use difference of squares.

$$ab = \frac{\sqrt{2}}{4}(\sqrt{3}+1)\frac{\sqrt{2}}{4}(\sqrt{3}-1) = \frac{2}{16}(3-1) = \frac{1}{4},$$
$$a^2 - b^2 = (a-b)(a+b) = \frac{\sqrt{2}}{4}(2) \cdot \frac{\sqrt{2}}{4}(2\sqrt{3}) = \frac{\sqrt{3}}{2}.$$

$$\mathbf{R}^2 = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sqrt{3} & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & \sqrt{3} \end{pmatrix}.$$

$k = \frac{1}{2}$. *(2)*

(c) The x-z block of $\mathbf{R}^2$ is a rotation through $30°$, so powers add this angle.

$$\mathbf{R}^6 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{R}^{12} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \mathbf{R}^{24} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The three matrices are as displayed. **(3)**

(d) The middle coordinate is fixed, so the axis is the y-axis. Since $\mathbf{R}^{24} = \mathbf{I}$, the angle is $360°/24 = 15°$. The signs in the x-z block give the clockwise sense.

A rotation through $15°$ clockwise about the y-axis. **(3)**

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