Maths with Matt

Physics and maths resources

OCR AS Level Further Mathematics A

H235/Y531 - Pure Core

Monday 13 May 2024

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the OCR Monday 13 May 2024 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

Use a matrix method to determine the solution of the following simultaneous equations.

$$\begin{aligned} 2x - 3y + z &= 1, \\ x - 2y - 4z &= 40, \\ 5x + 6y - z &= 61. \end{aligned}$$

Model solution

A matrix method writes all three equations as one matrix equation, then uses the inverse of the coefficient matrix to isolate the vector of unknowns.

$$\begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & -4 \\ 5 & 6 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 40 \\ 61 \end{pmatrix}.$$

$$\begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & -4 \\ 5 & 6 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 0.208 & 0.024 & 0.112 \\ -0.152 & -0.056 & 0.072 \\ 0.128 & -0.216 & -0.008 \end{pmatrix}.$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0.208 & 0.024 & 0.112 \\ -0.152 & -0.056 & 0.072 \\ 0.128 & -0.216 & -0.008 \end{pmatrix} \begin{pmatrix} 1 \\ 40 \\ 61 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \\ -9 \end{pmatrix}.$$

$x = 8,\ y = 2,\ z = -9.$ ***(4)***

Question 2

Question 2

In this question you must show detailed reasoning.

(a) Express $\dfrac{8+i}{2-i}$ in the form $a+bi$ where a and b are real.

(b) Solve the equation $4x^2-8x+5=0$. Give your answer(s) in the form $c+di$ where c and d are real.

Model solution

(a) To put a complex quotient into the required form, multiply the numerator and denominator by the conjugate of the denominator. This makes the denominator real.

$$\frac{8+i}{2-i}\times\frac{2+i}{2+i}=\frac{16+8i+2i+i^2}{4-i^2}=\frac{15+10i}{5}=3+2i.$$

$3+2i.$ ***(2)***

(b) The quadratic formula is appropriate because the equation does not factor into real linear factors. A negative discriminant will give complex roots.

$$x=\frac{8\pm\sqrt{(-8)^2-4(4)(5)}}{2(4)}=\frac{8\pm\sqrt{-16}}{8}=\frac{8\pm4i}{8}=1\pm\frac{1}{2}i.$$

$x=1+\frac{1}{2}i$ **or** $x=1-\frac{1}{2}i.$ ***(2)***

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Question 3

Question 3

(a) (i) Find

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix}.$$

(ii) State a geometrical relationship between the answer to part (a)(i) and the vectors $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix}$.

(iii) Verify the relationship stated in part (a)(ii).

(b) Find the angle between the vectors $2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $4\mathbf{i} - \mathbf{j} + 8\mathbf{k}$.

Model solution

(a)(i) Use the determinant form of the cross product to obtain a vector perpendicular to the two given vectors.

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 2(-2) - (-1)(5) \\ (-1)(3) - 1(-2) \\ 1(5) - 2(3) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}.$$

$\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$. *(1)*

(a)(ii) A cross product is perpendicular to each of the vectors used to form it.

$\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ *is perpendicular to both given vectors.* *(1)*

(a)(iii) A zero dot product verifies perpendicularity.

$$\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 1 - 2 + 1 = 0, \qquad \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} = 3 - 5 + 2 = 0.$$

Both dot products are zero, so the stated perpendicularity is verified. *(2)*

(b) For an angle between two vectors, use the scalar-product formula $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$.

$$\mathbf{a} \cdot \mathbf{b} = (2)(4) + (-2)(-1) + (1)(8) = 18,$$

$$|\mathbf{a}| = \sqrt{2^2 + (-2)^2 + 1^2} = 3, \qquad |\mathbf{b}| = \sqrt{4^2 + (-1)^2 + 8^2} = 9.$$

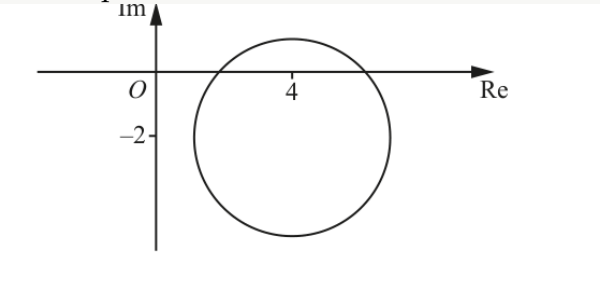
$$\cos\theta = \frac{18}{(3)(9)} = \frac{2}{3}, \qquad \theta = \cos^{-1}\left(\frac{2}{3}\right) = 48.2^\circ.$$

$\theta = 48.2^\circ$ *(to 1 decimal place).* *(3)*

Question 4

Question 4

The Argand diagram shows a circle of radius 3. The centre of the circle is the point which represents the complex number $4 - 2i$.



(a) Use set notation to define the locus of complex numbers, z, represented by points which lie on the circle.

The locus L is defined by

$$L = \{z : z \in \mathbb{C},\ |z - i| = |z + 2|\}.$$

(b) On the Argand diagram in the Printed Answer Booklet, sketch and label the locus L.

You are given that the locus

$$\left\{z : z \in \mathbb{C},\ \arg(z - 1) = \frac{1}{4}\pi,\ \text{Re}(z) = 3\right\}$$

contains only one number.

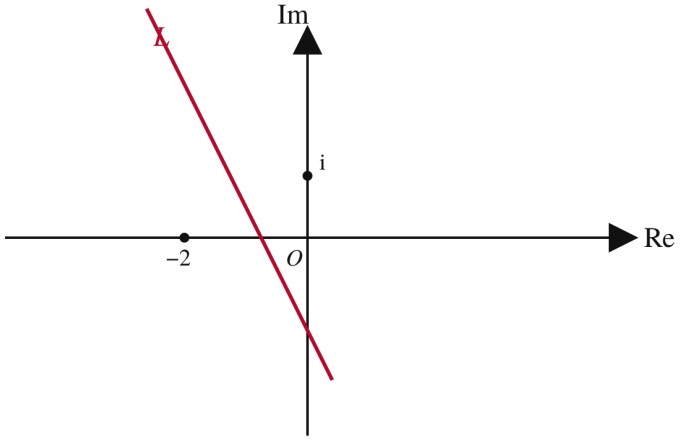
(c) Find this number.

Model solution

(a) A point is on a circle precisely when its distance from the centre is the radius. In complex notation the distance from $4 - 2i$ is $|z - (4 - 2i)|$.

$\{z : z \in \mathbb{C},\ |z - (4 - 2i)| = 3\}$. *(2)*

(b) The definition of L says that points are equidistant from i and -2. Therefore L is the perpendicular bisector of the line segment joining those two points.



The perpendicular bisector of the segment joining i and -2, labelled L. *(2)*

(c) The condition $\mathrm{Re}(z) = 3$ *gives* $x = 3$*. Also,* $\arg(z-1) = \frac{1}{4}\pi$ *means the line from* $1 + 0i$ *has gradient* $\tan(\frac{1}{4}\pi) = 1$*, so* $y = x - 1$*.*

$$y = 3 - 1 = 2.$$

$z = 3 + 2i.$ **(2)**

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Question 5

Question 5

The line through points $A(8, -7, -2)$ and $B(11, -9, 0)$ is denoted by L_1.
(a) Find a vector equation for L_1.
(b) Determine whether the point $(26, -19, -14)$ lies on L_1.
The line L_2 passes through the origin, O, and intersects L_1 at the point C. The lines L_1 and L_2 are perpendicular.
(c) By using the fact that C lies on L_1, find a vector equation for L_2.
(d) Hence find the shortest distance from O to L_1.

Model solution

(a) A direction vector for a line through two points is found by subtracting their position vectors.

$$\overrightarrow{AB} = \begin{pmatrix} 11 \\ -9 \\ 0 \end{pmatrix} - \begin{pmatrix} 8 \\ -7 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}.$$

$\mathbf{r} = \begin{pmatrix} 8 \\ -7 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$, *where* $\lambda \in \mathbb{R}$. *(2)*

(b) Substitute the coordinates of the proposed point into the three component equations. A single parameter value must work in every component.

$$8 + 3\lambda = 26 \implies \lambda = 6, \qquad -7 - 2\lambda = -19 \implies \lambda = 6,$$

$$-2 + 2\lambda = -14 \implies \lambda = -6.$$

The parameter values are inconsistent, so $(26, -19, -14)$ *does not lie on* L_1. *(2)*

(c) Write the position vector of C *using* L_1. *Since* OC *lies on* L_2 *and* $L_2 \perp L_1$, *this vector must have zero dot product with the direction vector of* L_1.

$$\overrightarrow{OC} = \begin{pmatrix} 8 \\ -7 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 + 3\lambda \\ -7 - 2\lambda \\ -2 + 2\lambda \end{pmatrix}.$$

$$\begin{pmatrix} 8 + 3\lambda \\ -7 - 2\lambda \\ -2 + 2\lambda \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} = 0.$$

$$3(8 + 3\lambda) - 2(-7 - 2\lambda) + 2(-2 + 2\lambda) = 0$$

$$34 + 17\lambda = 0 \qquad \implies \qquad \lambda = -2.$$

$$\overrightarrow{OC} = \begin{pmatrix} 8 \\ -7 \\ -2 \end{pmatrix} - 2 \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ -6 \end{pmatrix}.$$

$\mathbf{r} = \mu \begin{pmatrix} 2 \\ -3 \\ -6 \end{pmatrix}$, *where* $\mu \in \mathbb{R}$. *(4)*

(d) The perpendicular from a point to a line is the shortest route, so the required distance is $|\overrightarrow{OC}|$.

$$|OC| = \sqrt{2^2 + (-3)^2 + (-6)^2} = \sqrt{49} = 7.$$

7 units. **(2)**

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Question 6

Question 6

You are given that

$$A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

where a is a constant.
Prove by induction that

$$A^n = \begin{pmatrix} 1 & an \\ 0 & 1 \end{pmatrix}$$

for all integers $n \geq 1$.

Model solution

Use induction because the statement is claimed for every positive integer n. First verify the base case, then show that truth for $n = k$ forces truth for $n = k + 1$.
For $n = 1$,

$$A^1 = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a(1) \\ 0 & 1 \end{pmatrix}.$$

So the result is true when $n = 1$.
Assume that it is true when $n = k$; that is, assume

$$A^k = \begin{pmatrix} 1 & ak \\ 0 & 1 \end{pmatrix}.$$

Then

$$\begin{aligned} A^{k+1} &= A^k A \\ &= \begin{pmatrix} 1 & ak \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & a + ak \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & a(k+1) \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Thus the result is true for $n = k + 1$ whenever it is true for $n = k$. Since it is true for $n = 1$, it is true for all integers $n \geq 1$.

$A^n = \begin{pmatrix} 1 & an \\ 0 & 1 \end{pmatrix}$ *for all integers* $n \geq 1$. **(5)**

Question 7

Question 7

In this question you must show detailed reasoning.
The roots of the equation

$$2x^3 - 3x^2 - 3x + 5 = 0$$

are α, β and γ.
By considering $(\alpha+\beta+\gamma)^2$ and $(\alpha\beta+\beta\gamma+\gamma\alpha)^2$, determine a cubic equation with integer coefficients whose roots are

$$\frac{\alpha\beta}{\gamma}, \qquad \frac{\beta\gamma}{\alpha}, \qquad \frac{\gamma\alpha}{\beta}.$$

Model solution

Vieta's formulae give the symmetric sums of the original roots. These are exactly what is needed to form the sum, pairwise sum and product of the new roots.

$$\alpha + \beta + \gamma = \frac{3}{2}, \qquad \alpha\beta + \beta\gamma + \gamma\alpha = -\frac{3}{2}, \qquad \alpha\beta\gamma = -\frac{5}{2}.$$

Let the new roots be $p = \dfrac{\alpha\beta}{\gamma}$, $q = \dfrac{\beta\gamma}{\alpha}$ *and* $r = \dfrac{\gamma\alpha}{\beta}$. *First,*

$$pqr = \frac{\alpha\beta}{\gamma} \cdot \frac{\beta\gamma}{\alpha} \cdot \frac{\gamma\alpha}{\beta} = \alpha\beta\gamma = -\frac{5}{2}.$$

Squaring the pairwise sum of the original roots gives

$$(\alpha\beta + \beta\gamma + \gamma\alpha)^2 = \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 + 2\alpha\beta\gamma(\alpha + \beta + \gamma),$$

$$\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = \left(-\frac{3}{2}\right)^2 - 2\left(-\frac{5}{2}\right)\left(\frac{3}{2}\right) = \frac{39}{4}.$$

$$p + q + r = \frac{\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2}{\alpha\beta\gamma} = \frac{39/4}{-5/2} = -\frac{39}{10}.$$

Also, $pq = \beta^2$, $qr = \gamma^2$ *and* $rp = \alpha^2$. *Squaring the sum of the original roots therefore gives*

$$pq + qr + rp = \alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) = \frac{21}{4}.$$

A cubic with roots p, q, r *is* $x^3 - (p + q + r)x^2 + (pq + qr + rp)x - pqr = 0$.

$$x^3 + \frac{39}{10}x^2 + \frac{21}{4}x + \frac{5}{2} = 0.$$

$20x^3 + 78x^2 + 105x + 50 = 0.$ **(6)**

Question 8

Question 8

Three transformations, T_A, T_B and T_C, are represented by the matrices A, B and C respectively.
You are given that

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Find the matrix which represents the inverse transformation of T_A.
(b) By considering matrix multiplication, determine whether T_A followed by T_B is the same transformation as T_B followed by T_A.
Transformations R and S are each defined as being the result of successive transformations, as specified in the table.

Transformation	First transformation	followed by
R	T_A followed by T_B	T_C
S	T_A	T_B followed by T_C

(c) Explain, using a property of matrix multiplication, why R and S are the same transformations.
A quadrilateral, Q, has vertices D, E, F and G in anticlockwise order from D. Under transformation R, Q's image, Q', has vertices D', E', F' and G' (where D' is the image of D, etc). The area of Q, in suitable units, is 5.
You are given that $\det C = a^2 + 1$ where a is a real constant.
(d) (i) Determine the order of the vertices of Q', starting anticlockwise from D'.
(ii) Find, in terms of a, the area of Q'.
(iii) Explain whether the inverse transformation for R exists. Justify your answer.

Model solution

(a) Invert the 2×2 matrix using its determinant, which is 3.

$$A^{-1} = \frac{1}{3}\begin{pmatrix} 3 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

$\begin{pmatrix} 1 & 0 \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$. *(1)*

(b) With column vectors, applying T_A then T_B has matrix BA, whereas applying T_B then T_A has matrix AB. Calculate both to compare them.

$$AB = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & -3 \end{pmatrix},$$

$$BA = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & -3 \end{pmatrix}.$$

No: $BA \neq AB$, so the transformations are not the same. (2)

(c) The order of application gives the matrix $C(BA)$ for R, and $(CB)A$ for S. Matrix multiplication is associative, so regrouping does not change the product.

$$C(BA) = (CB)A.$$

R and S are the same transformation by associativity of matrix multiplication. (2)

(d)(i) A negative determinant reverses orientation. The determinant of R is

$$\det(CBA) = \det C \det B \det A = (a^2+1)(-1)(3) = -3(a^2+1) < 0,$$

because $a^2+1 > 0$ for real a. Thus the anticlockwise order is reversed.

D', G', F', E'. **(2)**

(d)(ii) Areas are multiplied by the absolute value of the determinant.

$$\text{area}(Q') = 5\left|-3(a^2+1)\right| = 15(a^2+1).$$

$15(a^2+1)$ *square units.* **(1)**

(d)(iii) A transformation has an inverse exactly when its matrix has non-zero determinant. Here $-3(a^2+1)$ is never zero for real a.

Yes, the inverse transformation for R exists. **(2)**

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Question 9

Question 9

In this question you must show detailed reasoning.
You are given that a is a real root of the equation

$$x^4 + x^3 + 3x^2 - 5x = 0.$$

You are also given that $a + 2 + 3i$ is one root of the equation

$$z^4 - 2(1+a)z^3 + (21a - 10)z^2 + (86 - 80a)z + (285a - 195) = 0.$$

Determine all possible values of z.

Model solution

First find every real value of a, since each possible value produces a different quartic in z.

$$x^4 + x^3 + 3x^2 - 5x = x(x^3 + x^2 + 3x - 5).$$

$$1^3 + 1^2 + 3(1) - 5 = 0,$$

so $x - 1$ is a factor of the cubic.

$$x^4 + x^3 + 3x^2 - 5x = x(x-1)(x^2 + 2x + 5).$$

$$\Delta = 2^2 - 4(1)(5) = -16 < 0,$$

so the quadratic has no real roots. Therefore

$$a = 0 \quad \textit{or} \quad a = 1.$$

If $a = 0$, the given root is $2 + 3i$, and because the quartic has real coefficients its conjugate $2 - 3i$ is also a root. Their quadratic factor is

$$(z - (2+3i))(z - (2-3i)) = z^2 - 4z + 13.$$

$$z^4 - 2z^3 - 10z^2 + 86z - 195 = (z^2 - 4z + 13)(z^2 + 2z - 15).$$

$$z^2 + 2z - 15 = (z+5)(z-3).$$

This case gives four roots:

$$z = 2 + 3i, \quad 2 - 3i, \quad -5, \quad 3.$$

If $a = 1$, the given root is $3 + 3i$, and its conjugate $3 - 3i$ is also a root. Their quadratic factor is

$$(z - (3+3i))(z - (3-3i)) = z^2 - 6z + 18.$$

$$z^4 - 4z^3 + 11z^2 + 6z + 90 = (z^2 - 6z + 18)(z^2 + 2z + 5).$$

$$z^2 + 2z + 5 = 0 \quad \Longrightarrow \quad z = -1 \pm 2i.$$

This case gives four roots:

$$z = 3 + 3i, \quad 3 - 3i, \quad -1 + 2i, \quad -1 - 2i.$$

$z \in \{2 + 3i,\ 2 - 3i,\ -5,\ 3,\ 3 + 3i,\ 3 - 3i,\ -1 + 2i,\ -1 - 2i\}.$ ***(8)***