Maths with Matt

Physics and maths resources

OCR AS Level Further Mathematics A

H235/Y531 - Pure Core

Monday 12 May 2025

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the OCR Monday 12 May 2025 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

(a) The complex number z is such that $|z| = 7$ and $\arg(z) = 2.2$ radians. Express z in cartesian form. [3]
(b) Use an algebraic method to determine the exact square roots of $1 + 4\sqrt{3}i$. [5]

Model solution

(a)
The modulus and argument give the polar form, so resolve the modulus into its real and imaginary components.

$$z = 7\cos(2.2) + 7\sin(2.2)i = -4.119507\ldots + 5.659475\ldots i.$$

$z = -4.12 + 5.66i$ *(3)*

(b)
Let one square root be $a + bi$, where a and b are real. Squaring this expression and comparing real and imaginary parts is the appropriate algebraic method.

$$(a + bi)^2 = a^2 - b^2 + 2abi = 1 + 4\sqrt{3}i.$$

$$a^2 - b^2 = 1, \qquad 2ab = 4\sqrt{3}.$$

Since $a \neq 0$, the second equation gives $b = 2\sqrt{3}/a$. Substitution removes b.

$$a^2 - \left(\frac{2\sqrt{3}}{a}\right)^2 = 1 \quad \Longrightarrow \quad a^4 - a^2 - 12 = 0.$$

$$(a^2 - 4)(a^2 + 3) = 0.$$

As a is real, $a^2 = 4$. Taking $a = 2$ gives $b = \sqrt{3}$; the other root is its negative.

The square roots are $2 + \sqrt{3}i$ and $-2 - \sqrt{3}i$. *(5)*

Question 2

Question 2

Two vectors **a** and **b** are given by

$$\mathbf{a} = \begin{pmatrix} 2 \\ -3 \\ 13 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} -4 \\ 6 \\ p \end{pmatrix},$$

where p is a constant.
(**a**) Find expressions in terms of p for each of $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$. [3]
(**b**) Hence or otherwise find the value of p in each of the following cases: **a** and **b** are perpendicular; **a** and **b** are parallel. [2]

Model solution

(a)
Use the component formula for a scalar product, and the determinant formula for a vector product.

$$\mathbf{a} \cdot \mathbf{b} = 2(-4) + (-3)(6) + 13p = 13p - 26.$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 13 \\ -4 & 6 & p \end{vmatrix} = \begin{pmatrix} -3p - 78 \\ -2p - 52 \\ 0 \end{pmatrix}.$$

$\mathbf{a} \cdot \mathbf{b} = 13p - 26, \quad \mathbf{a} \times \mathbf{b} = \begin{pmatrix} -3p - 78 \\ -2p - 52 \\ 0 \end{pmatrix}.$ *(3)*

(b)
Perpendicular vectors have scalar product zero. Parallel vectors have vector product zero.

$$13p - 26 = 0 \implies p = 2 \quad \textit{(perpendicular)}.$$

$$-3p - 78 = 0 \implies p = -26,$$

and the second component is then $-2(-26) - 52 = 0$*, confirming parallelism.*

Perpendicular: $p = 2$. *Parallel:* $p = -26$. *(2)*

Question 3

Question 3

The roots of the equation $x^2 + \frac{3}{2}x + \frac{5}{2} = 0$ are denoted by α and β.
(a) Write down the value of $\alpha + \beta$ and the value of $\alpha\beta$. [2]
(b) Using the answers to part **(a)** determine the value of each of $\alpha^2 + \beta^2$ and $\frac{1}{\alpha} + \frac{1}{\beta}$. [4]

Model solution

(a)
For $x^2 + bx + c = 0$, the sum and product of the roots are $-b$ and c, respectively.

$\alpha + \beta = -\frac{3}{2}, \quad \alpha\beta = \frac{5}{2}.$ *(2)*

(b)
The square of the sum gives $\alpha^2 + \beta^2$, while adding the reciprocals puts the known sum over the known product.

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \left(-\frac{3}{2}\right)^2 - 2\left(\frac{5}{2}\right) = -\frac{11}{4}.$$

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{-\frac{3}{2}}{\frac{5}{2}} = -\frac{3}{5}.$$

$\alpha^2 + \beta^2 = -\frac{11}{4}, \quad \frac{1}{\alpha} + \frac{1}{\beta} = -\frac{3}{5}.$ *(4)*

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Question 4

Question 4

Two transformations, T_A and T_B, are represented by matrices A and B respectively. The matrix A is given by

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(a) (i) Describe the transformation T_A. [1]
(ii) Explain geometrically why $A^{-1} = A$. [1]
The matrix B is given by

$$B = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}.$$

(b) Describe the transformation T_B. [2]
The transformation T_C is equivalent to T_A followed by T_B.
(c) Determine the single matrix which represents T_C. [2]

Model solution

(a)(i)
The matrix interchanges the x- and y-coordinates.

Reflection in the line $y = x$. *(1)*

(a)(ii)
A second reflection in the same line returns every point to its original position. Thus the inverse transformation is the same reflection.

$A^{-1} = A$. *(1)*

(b)
Compare the matrix with $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Here $\cos\theta = \frac{1}{2}$ and $\sin\theta = \frac{\sqrt{3}}{2}$.

Rotation through $\frac{\pi}{3}$ anticlockwise about the origin. *(2)*

(c)
The rightmost matrix acts first, so "T_A followed by T_B" is represented by BA.

$$BA = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}.$$

$\begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ *(2)*

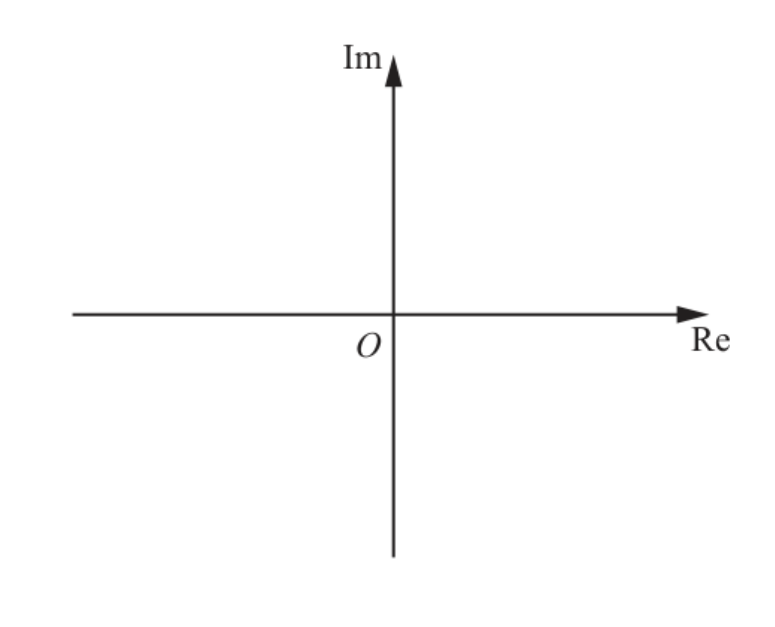
Question 5

Question 5

The locus L is defined by

$$L = \{z \in \mathbb{C} : |z - (20 + 15i)| \leq 7\}.$$

(**a**) On the Argand diagram below, sketch and label L. [2]



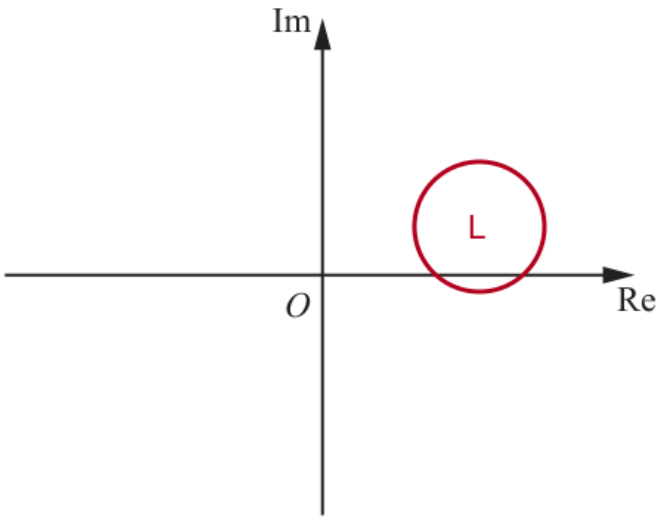
(**b**) Determine the value of $z \in L$ for which the value of $|z|$ is smallest. Give your answer in cartesian form. [3]

(**c**) Determine the largest value of $\arg(z)$ for $z \in L$. [3]

Model solution

(a)

A modulus inequality of this form describes the closed disc with centre $20 + 15i$ and radius 7.



L is the closed disc with centre $20 + 15i$ and radius 7. **(2)**

(b)

The closest point of a circle to the origin lies on the line from the origin to its centre. First find the centre's distance from the origin.

$$|20 + 15i| = \sqrt{20^2 + 15^2} = 25.$$

The required point is 7 units back from the centre towards the origin, so its modulus is $25 - 7 = 18$. Scale the centre vector to length 18.

$$z = \frac{18}{25}(20 + 15i) = \frac{72}{5} + \frac{54}{5}i.$$

$z = \frac{72}{5} + \frac{54}{5}i$. **(3)**

(c)

The largest argument is given by the upper tangent from the origin to the circle. The centre makes angle $\tan^{-1}(15/20)$ with the real axis, and the tangent makes an additional angle whose sine is $7/25$.

$$\max \arg(z) = \tan^{-1}\left(\frac{3}{4}\right) + \sin^{-1}\left(\frac{7}{25}\right) = 0.927\ldots \textit{ radians}.$$

The largest value is 0.927 radians. **(3)**

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Question 6

Question 6

The equations of two lines, l_1 and l_2, are

$$l_1 : \mathbf{r} = \begin{pmatrix} 16 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -27 \\ -19 \end{pmatrix} \quad \text{and} \quad l_2 : \mathbf{r} = \begin{pmatrix} 3 \\ 10 \\ -10 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 10 \\ 10 \end{pmatrix}.$$

(**a**) Show that l_1 and l_2 intersect at a single point, P, giving the coordinates of P. [5]
O is the origin of the coordinate system. The point Q lies on the line segment OP.
(**b**) Comment on the claim that the distance OQ is less than 100. [2]

Model solution

(a)

At an intersection the three corresponding coordinates are equal. This gives simultaneous equations in the two line parameters.

$$16 + 2\lambda = 3 + \mu, \qquad -1 - 27\lambda = 10 + 10\mu, \qquad 3 - 19\lambda = -10 + 10\mu.$$

Subtracting the second equation from the third eliminates μ.

$$4 + 8\lambda = -20 \implies \lambda = -3.$$

$$16 + 2(-3) = 3 + \mu \implies \mu = 7.$$

The unused coordinate equation checks the result: $3 - 19(-3) = 60$ and $-10 + 10(7) = 60$. Thus the same parameter pair satisfies all three equations, giving one common point.

$$\mathbf{OP} = \begin{pmatrix} 16 \\ -1 \\ 3 \end{pmatrix} - 3 \begin{pmatrix} 2 \\ -27 \\ -19 \end{pmatrix} = \begin{pmatrix} 10 \\ 80 \\ 60 \end{pmatrix}.$$

The lines intersect once at $P = (10, 80, 60)$. **(5)**

(b)

Because Q is on the segment OP, its distance from O can be as large as OP.

$$OP = \sqrt{10^2 + 80^2 + 60^2} = \sqrt{10100} = 100.499\ldots$$

Writing $\mathbf{OQ} = \nu\mathbf{OP}$, where $0 \leq \nu \leq 1$, shows that $OQ = \nu OP$. Therefore the claim is usually true for points on the segment, but it is not guaranteed: for $\nu > 100/\sqrt{10100} = 0.995037\ldots$, the distance exceeds 100.

The claim is not certain; OQ can exceed 100. **(2)**

Question 7

Question 7

Prove by induction that $n! > 20^n$ for all integers $n \geq 52$. [5]

Model solution

Use induction starting at $n = 52$, as this is the first integer required.

$$52! = 8.07\ldots \times 10^{67} \quad \textit{and} \quad 20^{52} = 4.50\ldots \times 10^{67}.$$

Hence $52! > 20^{52}$, so the statement is true in the base case.
Now assume that the statement is true for an integer $k \geq 52$; that is, assume $k! > 20^k$. Then

$$\begin{aligned}(k+1)! &= (k+1)k! \\ &> (k+1)20^k \\ &> 20 \cdot 20^k \qquad \textit{since } k+1 > 20 \\ &= 20^{k+1}.\end{aligned}$$

Thus truth for $n = k$ implies truth for $n = k+1$. Since the base case is true, induction proves the result for every integer $n \geq 52$.

$n! > 20^n$ *for all integers* $n \geq 52$. *(5)*

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Question 8

Question 8

Matrix A is given by

$$A = \begin{pmatrix} 5+a & 5 & 1 \\ 3 & 13+a & 6 \\ a-4 & -20 & -9 \end{pmatrix},$$

where a is a constant and all entries of A are integers.
The transformation represented by A is applied to a shape of volume 8 units. The image shape has volume 40 units and the orientation of the image is reversed.
Determine the image under A of the point $(1, 2, 3)$. [7]

Model solution

In three dimensions, the magnitude of the determinant is the volume scale factor. Reversal of orientation makes the determinant negative, so $\det A = -40/8 = -5$.

$$\begin{aligned} \det A &= (a+5)\big((13+a)(-9) - 6(-20)\big) \\ &\quad - 5\big(3(-9) - 6(a-4)\big) + \big(3(-20) - (13+a)(a-4)\big) \\ &= -10a^2 - 21a + 22. \end{aligned}$$

$$-10a^2 - 21a + 22 = -5 \implies 10a^2 + 21a - 27 = 0 \implies (10a-9)(a+3) = 0.$$

Thus $a = \frac{9}{10}$ *or* $a = -3$. *The entries must be integers, so* $a = -3$.

$$A = \begin{pmatrix} 2 & 5 & 1 \\ 3 & 10 & 6 \\ -7 & -20 & -9 \end{pmatrix}.$$

Multiply the transformation matrix by the position vector of the point.

$$\begin{pmatrix} 2 & 5 & 1 \\ 3 & 10 & 6 \\ -7 & -20 & -9 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 15 \\ 41 \\ -74 \end{pmatrix}.$$

The image is $(15, 41, -74)$. *(7)*

Question 9

Question 9

(a) Prove that if z is any complex number then $zz^* = |z|^2$. [1]
(b) Prove that if w and z are any complex numbers then $(wz)^* = w^*z^*$. [2]
The complex number v is defined by

$$v = (10 + 3i)(9 + 4i).$$

(c) In this question you must show detailed reasoning. Express v in the form $a + bi$ where a and b are real. [2]
(d) In this question you must show detailed reasoning. By considering vv^*, write 10573 as the product of two prime factors. [3]

Model solution

(a)
Write a general complex number in cartesian form and multiply it by its conjugate.

$$z = a + bi \quad \Longrightarrow \quad zz^* = (a + bi)(a - bi) = a^2 + b^2 = |z|^2.$$

$zz^* = |z|^2$. ***(1)***

(b)
Let $w = a + bi$ and $z = c + di$, where all four letters are real. Calculate both sides.

$$(wz)^* = \big(ac - bd + (ad + bc)i\big)^* = ac - bd - (ad + bc)i.$$

$$w^*z^* = (a - bi)(c - di) = ac - bd - (ad + bc)i.$$

Both sides are equal, proving the required identity.

$(wz)^* = w^*z^*$. ***(2)***

(c)
Expand the brackets and use $i^2 = -1$.

$$\begin{aligned} v &= (10 + 3i)(9 + 4i) \\ &= 90 + 40i + 27i + 12i^2 \\ &= 78 + 67i. \end{aligned}$$

$v = 78 + 67i$. ***(2)***

(d)
Part (a) changes vv^* into the square of the modulus. Using the factorised definition of v and the conjugate product identity exposes integer factors.

$$vv^* = |v|^2 = 78^2 + 67^2 = 10573.$$

$$\begin{aligned} vv^* &= (10 + 3i)(9 + 4i)(10 - 3i)(9 - 4i) \\ &= (10 + 3i)(10 - 3i)(9 + 4i)(9 - 4i) \\ &= (100 + 9)(81 + 16) = 109 \times 97. \end{aligned}$$

$10573 = 109 \times 97$. ***(3)***