Maths with Matt

Physics and maths resources

OCR AS Level Further Mathematics A

H235/Y531 - Pure Core

Monday 4 October 2021

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the OCR Monday 4 October 2021 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

The lines l_1 and l_2 have the following equations.

$$l_1 : \mathbf{r} = \begin{pmatrix} 8 \\ -11 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 5 \\ 3 \end{pmatrix}, \qquad l_2 : \mathbf{r} = \begin{pmatrix} -6 \\ 11 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix}.$$

(a) Show that l_1 and l_2 intersect.
(b) Write down the point of intersection of l_1 and l_2.

Model solution

(a) At an intersection the position vectors are equal, so equate their components.

$$8 - 2\lambda = -6 - 3\mu, \qquad -11 + 5\lambda = 11 + \mu.$$

$$-2\lambda + 3\mu = -14, \qquad 5\lambda - \mu = 22.$$

Multiply the second equation by 3, then add it to the first to eliminate μ.

$$-2\lambda + 3\mu + 15\lambda - 3\mu = -14 + 66, \qquad 13\lambda = 52, \qquad \lambda = 4.$$

$$5(4) - \mu = 22 \quad \Rightarrow \quad \mu = -2.$$

The remaining components agree: $-2 + 3(4) = 10$ and $8 - (-2) = 10$. Thus the same point lies on both lines.

The lines intersect, with $\lambda = 4$ and $\mu = -2$. ***(4)***

(b) Substitute either parameter value into its line equation.

$$\begin{pmatrix} 8 \\ -11 \\ -2 \end{pmatrix} + 4 \begin{pmatrix} -2 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 9 \\ 10 \end{pmatrix}.$$

The point of intersection is $(0, 9, 10)$. ***(1)***

Question 2

Question 2

The equation $2x^3 + 3x^2 - 2x + 5 = 0$ has roots α, β and γ.
Use a substitution to find a cubic equation with integer coefficients whose roots are $\alpha + 1$, $\beta + 1$ and $\gamma + 1$.

Model solution

Each required root is one more than an original root, so set $u = x + 1$. Then $x = u - 1$, and the new equation in u has exactly the required roots.

$$2(u-1)^3 + 3(u-1)^2 - 2(u-1) + 5 = 0.$$

$$2(u^3 - 3u^2 + 3u - 1) + 3(u^2 - 2u + 1) - 2u + 2 + 5 = 0.$$

$$2u^3 - 6u^2 + 6u - 2 + 3u^2 - 6u + 3 - 2u + 2 + 5 = 0,$$

$$2u^3 - 3u^2 - 2u + 8 = 0.$$

A suitable equation is $2u^3 - 3u^2 - 2u + 8 = 0$. ***(4)***

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Question 3

Question 3

In this question you must show detailed reasoning.
The equation $x^4 - 7x^3 - 2x^2 + 218x - 1428 = 0$ has a root $3 - 5i$.
Find the other three roots of this equation.

Model solution

The polynomial has real coefficients. Therefore non-real roots occur in conjugate pairs, so $3 + 5i$ *is also a root.*

$$\big(x - (3 - 5i)\big)\big(x - (3 + 5i)\big) = x^2 - 6x + (3^2 + 5^2) = x^2 - 6x + 34.$$

This quadratic is a factor of the given quartic. Factoring the remaining quadratic gives

$$x^4 - 7x^3 - 2x^2 + 218x - 1428 = (x^2 - 6x + 34)(x^2 - x - 42).$$

$$x^2 - x - 42 = (x - 7)(x + 6).$$

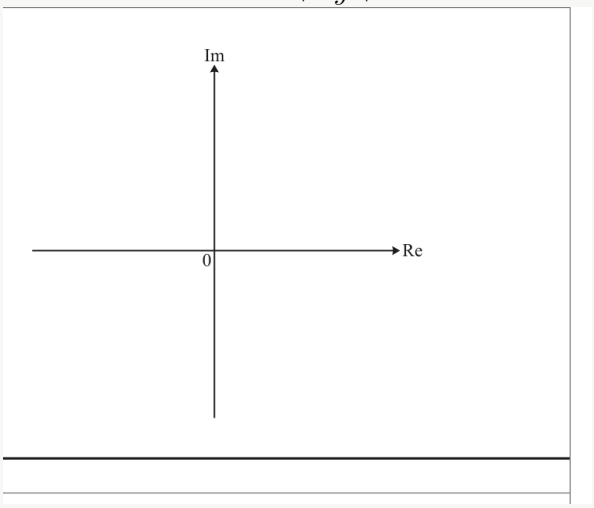
The other three roots are $3 + 5i$, 7 *and* -6. *(6)*

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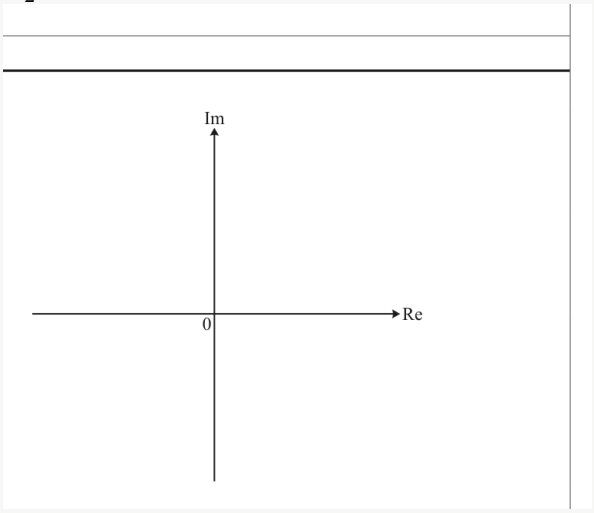
Question 4

Question 4

(a) A locus C_1 is defined by $C_1 = \{z : |z + i| \leq |z - 2|\}$.
(i) Indicate by shading on the Argand diagram in the Printed Answer Booklet the region representing C_1.
(ii) Find the cartesian equation of the boundary line of the region representing C_1, giving your answer in the form $ax + by + c = 0$.



(b) A locus C_2 is defined by $C_2 = \{z : |z + 1| \leq 3\} \cap \{z : |z - 2i| \geq 2\}$.
Indicate by shading on the Argand diagram in the Printed Answer Booklet the region representing C_2.



Model solution

(a)(i) The two moduli are distances from $(0, -1)$ and $(2, 0)$. The locus consists of points at least as close to $(0, -1)$ as to $(2, 0)$, so it is the appropriate side of their perpendicular bisector, including the boundary.

Shade the half-plane below the perpendicular bisector of $(0, -1)$ and $(2, 0)$, including the boundary. *(2)*

(a)(ii) Let $z = x + iy$. Squaring the two non-negative distances finds the boundary without square roots.

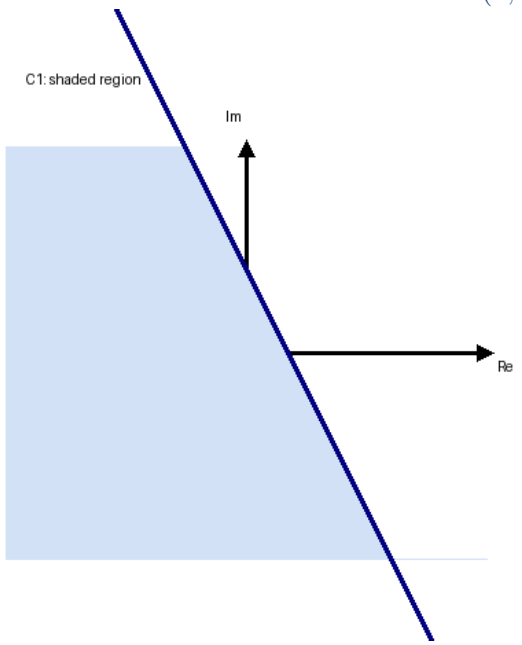
$$|z + i|^2 = x^2 + (y + 1)^2, qquad|z - 2|^2 = (x - 2)^2 + y^2.$$

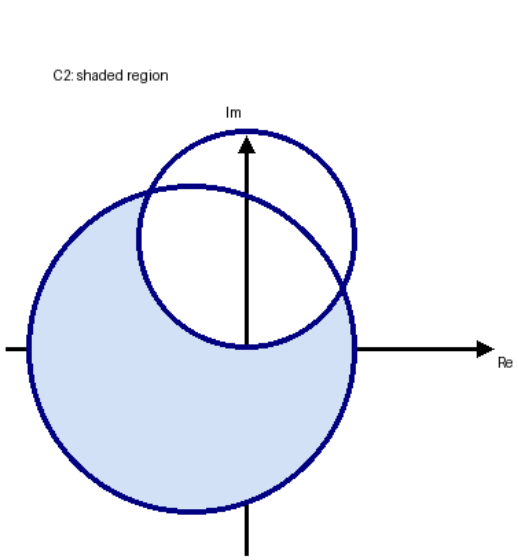
$$x^2 + y^2 + 2y + 1 = x^2 - 4x + 4 + y^2 \quad \Rightarrow \quad 4x + 2y - 3 = 0.$$

For example, $(0, 0)$ satisfies the locus inequality, so the required side is $4x + 2y - 3 \leq 0$.

The boundary is $4x + 2y - 3 = 0$. **(2)**

(b) The first condition is the closed disc with centre $(-1, 0)$ and radius 3. The second condition means on or outside the circle with centre $(0, 2)$ and radius 2. Their intersection is shaded below.





Shade the region inside (including the boundary of) the circle centred at $(-1, 0)$ of radius 3, but outside (including the boundary of) the circle centred at $(0, 2)$ of radius 2. **(3)**

Question 5

Question 5

Matrices **A** and **B** are given by

$$\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} \frac{5}{13} & -\frac{12}{13} \\ \frac{12}{13} & \frac{5}{13} \end{pmatrix}.$$

(a) Use **A** and **B** to disprove the proposition: "Matrix multiplication is commutative".
Matrix **B** represents the transformation T_B.
(b) Describe the transformation T_B.
(c) By considering the inverse transformation of T_B, determine $\mathbf{B}^{-1}$.
Matrix $\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}$ represents the transformation T_C. The transformation T_{BC} is transformation T_C followed by transformation T_B. An object shape of area 5 is transformed by T_{BC} to an image shape N.
(d) Determine the area of N.

Model solution

(a) A single counterexample disproves commutativity, so calculate the products in both orders.

$$\mathbf{AB} = \begin{pmatrix} -\frac{5}{13} & \frac{12}{13} \\ \frac{12}{13} & \frac{5}{13} \end{pmatrix}, \qquad \mathbf{BA} = \begin{pmatrix} -\frac{5}{13} & -\frac{12}{13} \\ -\frac{12}{13} & \frac{5}{13} \end{pmatrix}.$$

$\mathbf{AB} \neq \mathbf{BA}$, *so matrix multiplication is not commutative.* *(2)*

(b) Compare **B** *with the standard rotation matrix* $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

$$\cos\theta = \frac{5}{13}, \qquad \sin\theta = \frac{12}{13}, \qquad \theta = \tan^{-1}\left(\frac{12}{5}\right) \approx 67.4°.$$

T_B *is a rotation of* $67.4°$ *anticlockwise about the origin.* *(2)*

(c) The inverse rotation has the opposite angle, so the signs of the sine entries reverse.

$$\mathbf{B}^{-1} = \begin{pmatrix} \frac{5}{13} & \frac{12}{13} \\ -\frac{12}{13} & \frac{5}{13} \end{pmatrix}. \qquad \textit{(2)}$$

(d) Area is multiplied by the absolute value of the determinant. A rotation preserves area, while T_C *has area scale factor* $|\det \mathbf{C}| = 3$.

$$A_N = 5 \times 3 \times 1 = 15.$$

The area of N *is* 15. *(2)*

Question 6

Question 6

In this question you must show detailed reasoning.
(a) Solve the equation $2z^2 - 10z + 25 = 0$, giving your answers in the form $a + bi$.
(b) Solve the equation $3\omega - 2 = i(5 + 2\omega)$, giving your answer in the form $a + bi$.

Model solution

(a) The discriminant is negative, so use the quadratic formula and write the square root of -1 as i.

$$z = \frac{10 \pm \sqrt{(-10)^2 - 4(2)(25)}}{2(2)} = \frac{10 \pm \sqrt{-100}}{4} = \frac{5 \pm 5i}{2}.$$

$z = \frac{5}{2} + \frac{5}{2}i$ *or* $z = \frac{5}{2} - \frac{5}{2}i$. *(2)*

(b) First collect the terms containing ω, then divide by a complex number. Multiplying by the conjugate gives standard $a + bi$ form.

$$3\omega - 2 = 5i + 2i\omega \quad \Rightarrow \quad (3 - 2i)\omega = 2 + 5i,$$

$$\omega = \frac{2 + 5i}{3 - 2i} \times \frac{3 + 2i}{3 + 2i} = \frac{6 + 4i + 15i + 10i^2}{9 + 4} = \frac{-4 + 19i}{13}.$$

$\omega = -\frac{4}{13} + \frac{19}{13}i$. *(4)*

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Question 7

Question 7

Prove that $2^{3n} - 3^n$ is divisible by 5 for all integers $n \geq 1$.

Model solution

Use mathematical induction because the statement is to be proved for every positive integer n.
Base case: when $n = 1$,

$$2^{3(1)} - 3^1 = 8 - 3 = 5,$$

which is divisible by 5.
Inductive hypothesis: assume that for $n = k$, $2^{3k} - 3^k = 5p$ for some integer p.
For $n = k + 1$, use the index laws and the hypothesis to replace 2^{3k}.

$$\begin{aligned} 2^{3(k+1)} - 3^{k+1} &= 2^3 \cdot 2^{3k} - 3 \cdot 3^k \\ &= 8(5p + 3^k) - 3 \cdot 3^k \\ &= 40p + 5 \cdot 3^k \\ &= 5(8p + 3^k). \end{aligned}$$

Since $8p + 3^k$ is an integer, this is divisible by 5. Thus the result for k implies the result for $k + 1$.

The statement is true for $n = 1$ and the inductive step is true; therefore $2^{3n} - 3^n$ is divisible by 5 for all integers $n \geq 1$. ***(5)***

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Question 8

Question 8

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} t-1 & t-1 & t-1 \\ 1-t & 6 & t \\ 2-2t & 2-2t & 1 \end{pmatrix}.$$

(a) Find, in fully factorised form, an expression for $\det \mathbf{A}$ in terms of t.
(b) State the values of t for which $\mathbf{A}$ is singular.
You are given the following system of equations in x, y and z, where b is a real number.

$$\begin{aligned} (b^2+1)x + (b^2+1)y + (b^2+1)z &= 5, \\ (-b^2-1)x + 6y + (b^2+2)z &= 10, \\ (-2b^2-2)x + (-2b^2-2)y + z &= 15. \end{aligned}$$

(c) Determine which one of the following statements about the solution of the equations is true.
- There is a unique solution for all values of b.
- There is a unique solution for some, but not all, values of b.
- There is no unique solution for any value of b.

Model solution

(a) Expand along the first row, whose entries have the common factor $t-1$, then factorise the resulting quadratic.

$$\begin{aligned} \det \mathbf{A} &= (t-1)\big(6 - t(2-2t)\big) \\ &\quad - (t-1)\big((1-t) - t(2-2t)\big) \\ &\quad + (t-1)\big((1-t)(2-2t) - 6(2-2t)\big) \\ &= (t-1)(2t^2+9t-5) \\ &= (t-1)(2t-1)(t+5). \end{aligned}$$

$\det \mathbf{A} = (t-1)(2t-1)(t+5)$. *(3)*

(b) A matrix is singular precisely when its determinant is zero.

$t = -5,\ \frac{1}{2},\ 1$. *(1)*

(c) The coefficient matrix of the system is $\mathbf{A}$ with $t = b^2+2$. Since b is real, $t \geq 2$, which can never equal -5, $\frac{1}{2}$, or 1. Hence its determinant is never zero.

There is a unique solution for all values of b. *(2)*

Question 9

Question 9

The points $P(3, 5, -21)$ and $Q(-1, 3, -16)$ are on the ceiling of a long straight underground tunnel. A ventilation shaft must be dug from the point M on the ceiling of the tunnel midway between P and Q to horizontal ground level (where the z-coordinate is 0). The ventilation shaft must be perpendicular to the tunnel.
The path of the ventilation shaft is modelled by the vector equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, where $\mathbf{a}$ is the position vector of M.
You are given that $\mathbf{b} = \begin{pmatrix} 1 \\ s \\ t \end{pmatrix}$, where s and t are real numbers.
(a) Show that $s = 2.5t - 2$.
(b) Show that at the point where the ventilation shaft reaches the ground $\lambda = \frac{c}{t}$, where c is a constant to be determined.
(c) Using the results in parts (a) and (b), determine the shortest possible length of the ventilation shaft.
(d) Explain what the fact that $\mathbf{b} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \neq \mathbf{0}$ means about the direction of the ventilation shaft.

Model solution

(a) A direction vector for the tunnel is $\overrightarrow{PQ}$. Perpendicular directions have zero scalar product.

$$\overrightarrow{PQ} = \begin{pmatrix} -1 \\ 3 \\ -16 \end{pmatrix} - \begin{pmatrix} 3 \\ 5 \\ -21 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \\ 5 \end{pmatrix}.$$

$$\begin{pmatrix} -4 \\ -2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ s \\ t \end{pmatrix} = 0 \quad \Rightarrow \quad -4 - 2s + 5t = 0 \quad \Rightarrow \quad s = 2.5t - 2.$$

$s = 2.5t - 2.$ **(3)**

(b) First find the midpoint, because this is the stated starting point of the shaft.

$$\mathbf{a} = \frac{1}{2}\left(\begin{pmatrix} 3 \\ 5 \\ -21 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ -16 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 4 \\ -18.5 \end{pmatrix}.$$

At ground level the z-coordinate of $\mathbf{r}$ is zero.

$$-18.5 + \lambda t = 0 \quad \Rightarrow \quad \lambda = \frac{18.5}{t}.$$

$\lambda = \frac{c}{t}$, *where* $c = 18.5$. **(3)**

(c) The shaft displacement is $\lambda\mathbf{b}$. Substitute the relations for s and λ, then minimise its squared length; this avoids differentiating a square root.

$$L^2 = \left\| \frac{18.5}{t} \begin{pmatrix} 1 \\ 2.5t - 2 \\ t \end{pmatrix} \right\|^2 = \frac{18.5^2}{t^2}\left(1 + (2.5t - 2)^2 + t^2\right).$$

$$L^2 = \frac{1369}{4}\left(\frac{5}{t^2} - \frac{10}{t} + 7.25\right).$$

$$\frac{\mathrm{d}(L^2)}{\mathrm{d}t} = \frac{1369}{4}\left(-\frac{10}{t^3} + \frac{10}{t^2}\right) = 0 \quad \Rightarrow \quad t = 1.$$

The second derivative is positive at $t = 1$, so this stationary value is a minimum. Then $s = 0.5$ and $\lambda = 18.5$.

$$L = 18.5\left\|\begin{pmatrix} 1 \\ 0.5 \\ 1 \end{pmatrix}\right\| = 18.5\sqrt{1 + 0.25 + 1} = 18.5(1.5) = 27.75.$$

The shortest possible shaft length is 27.75 units. **(6)**

(d) A non-zero cross product means the two direction vectors are not parallel.

The ventilation shaft is not vertical (not parallel to the z-axis). **(1)**

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