Maths with Matt

Physics and maths resources

OCR AS Level Further Mathematics A

H235/Y534 - Discrete Mathematics

Friday 26 May 2023

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the OCR Friday 26 May 2023 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

Jane wants to travel from home to the local town.
Jane can do this by train, by bus or by both train and bus.
(**a**) Give an example of a problem that Jane could be answering that would give a construction problem.
A website gives Jane all the possible buses and trains that she could use. Jane finds 7 possible ways to make the journey.

- 2 of the 7 journeys involve travelling by train for at least part of the journey.
- 6 of the 7 journeys involve travelling by bus for at least part of the journey.

(**b**) Use the inclusion-exclusion principle to find how many of the 7 journeys involve travelling by both train and bus.

Model solution

(a) A construction problem asks us to produce an actual object satisfying conditions, rather than simply decide whether one exists. Here the object is a journey.

Find a bus or train, or a combination of both, that Jane can use to travel from home to the local town. *(1)*

(b) The total of the train and bus counts double-counts journeys using both modes, so inclusion-exclusion corrects this. Let T and B be the sets of train and bus journeys.

$$n(T \cup B) = n(T) + n(B) - n(T \cap B)$$

$$7 = 2 + 6 - n(T \cap B), \qquad n(T \cap B) = 1.$$

There is 1 journey involving both train and bus. *(2)*

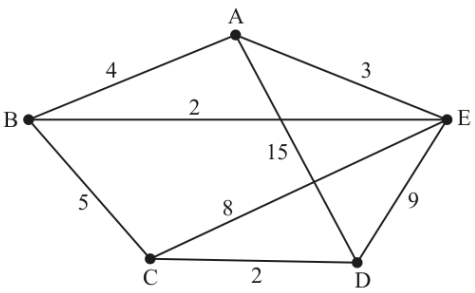
Question 2

Question 2

A network is shown below.

by both train and bus. [2

A network is shown below.



(a) Use an appropriate algorithm to find the least weight (shortest) path from A to D. [3

(b) Use Kruskal's algorithm to find a minimum spanning tree for the network. [3

(**a**) Use an appropriate algorithm to find the least weight (shortest) path from A to D.
(**b**) Use Kruskal's algorithm to find a minimum spanning tree for the network.

Model solution

(a) Because all arc weights are positive, Dijkstra's algorithm is appropriate. We label each vertex with its best currently known distance from A, updating a label only when a shorter route is found.

vertex made permanent	*new tentative labels*	*permanent distance*
A	$B = 4,\ E = 3,\ D = 15$	0
E	$B = 4,\ C = 11,\ D = 12$	3
B	$C = 9$	4
C	$D = 11$	9
D		11

The permanent label at D is 11. *Tracing its labels backwards gives* $D \leftarrow C \leftarrow B \leftarrow A$.

The shortest path is $A - B - C - D$, *of weight* $4 + 5 + 2 = 11$. *(3)*

(b) Kruskal's algorithm takes arcs in increasing order of weight, accepting an arc unless it would make a cycle.

arc considered	*decision*
$BE = 2$	*accept*
$CD = 2$	*accept*
$AE = 3$	*accept*
$AB = 4$	*reject: this would form* $A - E - B - A$
$BC = 5$	*accept*

Four accepted arcs connect all five vertices, so the spanning tree is complete.

A minimum spanning tree has arcs BE, CD, AE, BC, with total weight $2 + 2 + 3 + 5 = 12$. ***(3)***

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Question 3

Question 3

The list of numbers below is to be sorted into **increasing** order.

23 10 18 7 62 54 31 82

(**a**) Sort the list using bubble sort. You do not need to show intermediate working.
(**i**) Record the list that results at the end of each pass.
(**ii**) Record the number of swaps used in each pass.
(**b**) Now sort the original list using shuttle sort. You do not need to show intermediate working.
(**i**) Record the list that results at the end of each pass.
(**ii**) Record the number of swaps used in each pass.
(**c**) Using the total number of comparisons plus the total number of swaps as a measure of efficiency, explain why shuttle sort is more efficient for this particular list.
Bubble sort and shuttle sort are both $O(n^2)$.
(**d**) Explain what this means for the run-time of the algorithms when the length of the list being sorted changes from 1000 to 3000.

Model solution

(a)(i) Bubble sort repeatedly compares neighbouring items, moving the largest unsorted item to the right in each pass.

pass	*list at the end of the pass*							
1	10	18	7	23	54	31	62	82
2	10	7	18	23	31	54	62	82
3	7	10	18	23	31	54	62	82
4	7	10	18	23	31	54	62	82

The lists after the four passes are as shown. *(3)*

(a)(ii) Count each exchange of two neighbouring values during the corresponding pass.

pass	1	2	3	4
number of swaps	5	2	1	0

The numbers of swaps are $5,\ 2,\ 1,\ 0$. *(2)*

(b)(i) Shuttle sort inserts the next item into the already sorted part of the list, shuttling it left only as far as necessary.

pass	*list at the end of the pass*							
1	10	23	18	7	62	54	31	82
2	10	18	23	7	62	54	31	82
3	7	10	18	23	62	54	31	82
4	7	10	18	23	62	54	31	82
5	7	10	18	23	54	62	31	82
6	7	10	18	23	31	54	62	82
7	7	10	18	23	31	54	62	82

The lists after the seven passes are as shown. *(3)*

(b)(ii)

pass	1	2	3	4	5	6	7
number of swaps	1	1	3	0	1	2	0

The numbers of swaps are $1,\ 1,\ 3,\ 0,\ 1,\ 2,\ 0$. *(2)*

(c) Both methods make 8 *swaps, but comparisons also count in the stated measure. Bubble sort uses* $7+6+5+4=22$ *comparisons, whereas shuttle sort uses* $1+2+3+1+2+3+1=13$.

Shuttle sort is more efficient for this list because it makes fewer comparisons: 13 *rather than* 22*, while using the same number of swaps.* *(1)*

(d) The notation $O(n^2)$ *means the run-time is proportional to the square of the list length for large* n.

$$\frac{3000^2}{1000^2} = 3^2 = 9.$$

The run-time becomes approximately 9 *times as long.* *(1)*

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Question 4

Question 4

Graph G is a simply connected Eulerian graph with 4 vertices.
(a) (i) Explain why graph G cannot be a complete graph.
(ii) Determine the number of arcs in graph G, explaining your reasoning.
(iii) Show that graph G is a bipartite graph.
Graph H is a digraph with 4 vertices and no undirected arcs. The adjacency matrix below shows the number of arcs that directly connect each pair of vertices in digraph H.

From\To	A	B	C	D
A	0	1	0	1
B	0	0	2	0
C	2	1	0	1
D	0	1	1	0

(b) (i) Write down a feature of the adjacency matrix that shows that H has no loops.
(ii) Find the number of arcs in H.
(iii) Draw a possible digraph H.
(iv) Show that digraph H is semi-Eulerian by writing down a suitable trail.

Model solution

(a)(i) A complete graph on four vertices is K_4, so every vertex has degree 3. An Eulerian graph must have every vertex of even degree.

G cannot be complete: K_4 has vertices of odd degree 3, whereas every vertex of an Eulerian graph has even degree. *(2)*

(a)(ii) Since G is connected, no vertex has degree 0. Since it is simple with four vertices, no degree can exceed 3; since it is Eulerian, each degree is even. Thus every vertex has degree 2.

$$\textit{sum of degrees} = 4 \times 2 = 8 = 2 \times \textit{number of arcs}.$$

G has 4 arcs. *(2)*

(a)(iii) The graph is a 4-cycle. Put alternate vertices in different sets; every arc then joins one set to the other.

For vertices in cycle order v_1, v_2, v_3, v_4, the sets $\{v_1, v_3\}$ and $\{v_2, v_4\}$ form a bipartition. Hence G is bipartite. *(1)*

(b)(i) A diagonal entry counts arcs from a vertex back to itself.

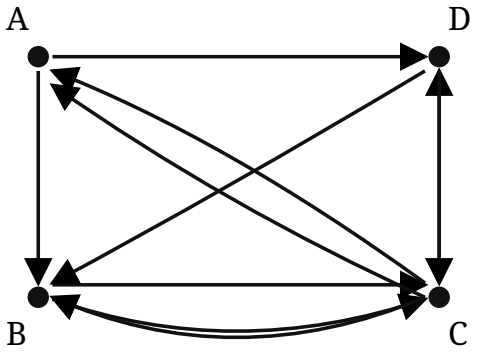
All entries on the leading diagonal are 0. *(1)*

(b)(ii) Each matrix entry counts directed arcs, so add all the entries.

$$(1+1)+2+(2+1+1)+(1+1) = 10.$$

There are 10 arcs in H. *(1)*

(b)(iii) The arrows in the diagram represent the entries of the matrix; parallel arrows represent the entry 2.



One possible digraph H is shown. **(1)**

(b)(iv) *A semi-Eulerian trail uses every directed arc exactly once and has different start and end vertices. Check each consecutive pair against the matrix.*

A suitable trail is $C-A-B-C-A-D-C-D-B-C-B$. It uses all 10 directed arcs once, starts at C and ends at B. **(2)**

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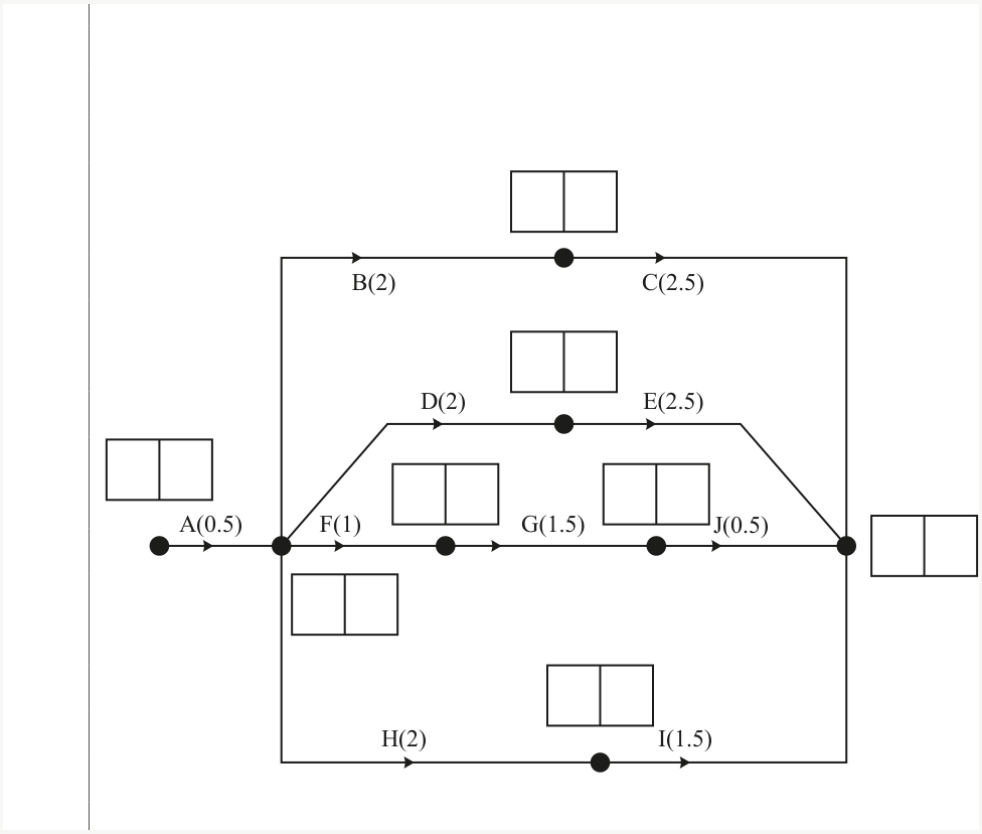
Question 5

Question 5

Hiro has been asked to organise a quiz.
The table below shows the activities involved, together with the immediate predecessors and the duration of each activity in hours.

	Activity	Immediate predecessors	Duration (hours)
A	Choose the topics	–	0.5
B	Find questions for round 1	A	2
C	Check answers for round 1	B	2.5
D	Find questions for round 2	A	2
E	Check answers for round 2	D	2.5
F	Choose pictures for picture round	A	1
G	Get permission to use pictures	F	1.5
H	Choose music for music round	A	2
I	Get permission to use music	H	1.5
J	Produce answer sheets	G	0.5

(**a**) A sketch of the activity network is provided in the Printed Answer Booklet. Apply a forward pass to determine the minimum project completion time.



(**b**) Use a backward pass to determine the critical activities. You can show your working on the activity network from part (a).
(**c**) Give the total float for each non-critical activity.
Hiro decides that there should be a final check of the answers which he will include as activity L. Activity L needs to be done after checking the answers for rounds 1 and 2 and also after getting permission to use the pictures and music but before producing the answer sheets.
(**d**) (**i**) Complete the activity network provided in the Printed Answer Booklet to show the new precedences, with the final check of the answers included as activity L.

(ii) As a result of including L, the minimum project completion time found in part (a) increases by 2.5 hours. Determine the duration of L.

Model solution

(a) A forward pass records the earliest time at which each event can occur. At a merge, take the larger incoming time, because every preceding activity must be complete.

event	start	A	B	C	D	E	F	G	H	I	J / finish
earliest time	0	0.5	2.5	5	2.5	5	1.5	3	2.5	4	5

The minimum project completion time is 5 hours. **(2)**

(b) For a backward pass, start from the completion time 5 and work backwards. At a split, take the smaller latest time, so that all following activities can still start on time.

event	start	A	B	C	D	E	F	G	H	I	J / finish
latest time	0	0.5	2.5	5	2.5	5	2	3.5	2.5	4	5

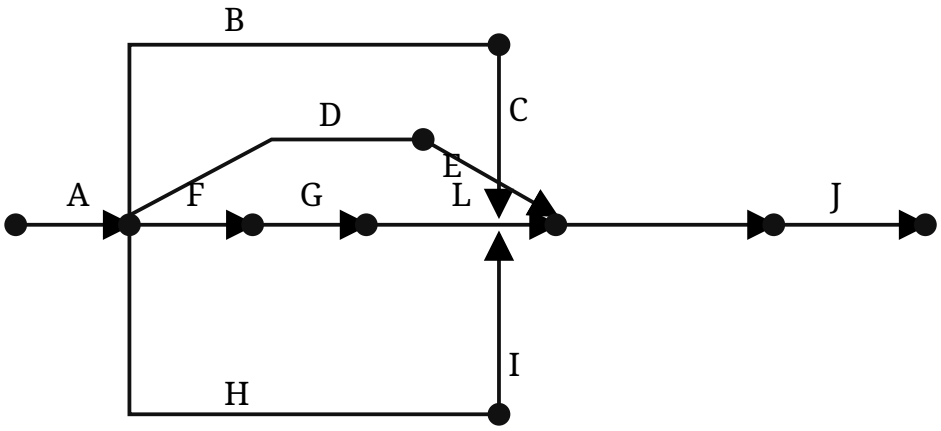
The critical activities are $A,\ B,\ C,\ D,\ E$. **(2)**

(c) The total float of an activity is latest finish − earliest start − duration.

activity	F	G	J	H	I
total float (hours)	1.5	1.5	1.5	1	1

F, G, J each have total float 1.5 hours; H, I each have total float 1 hour. **(3)**

(d)(i) The new activity must wait for C, E, G, I, then J must wait for it. The completed precedence network is shown below.



Activity L follows C, E, G, I and precedes J, as shown. **(2)**

(d)(ii) The original completion time is 5 hours, so the new completion time is 7.5 hours. The latest of the required predecessors of L finishes at time 5, and J then takes 0.5 hours.

$$5 + \text{duration of } L + 0.5 = 7.5, \qquad \text{duration of } L = 2.$$

The duration of L is 2 hours. **(2)**

Question 6

Question 6

Ryan and Casey are playing a card game in which they each have four cards.

- Ryan's cards have the letters A, B, C and D.
- Casey's cards have the letters W, X, Y and Z.

Each player chooses one of their four cards and they simultaneously reveal their choices.
The table shows the number of points won by Ryan for each combination of strategies.

	W	X	Y	Z
A	4	0	2	1
B	0	2	-3	4
C	1	4	-4	5
D	6	-1	5	0

For example, if Ryan chooses A and Casey chooses W then Ryan wins 4 points (and Casey loses 4 points).
Both Ryan and Casey are trying to win as many points as possible.
(a) Use dominance to reduce the 4×4 table for the zero-sum game above to a 4×2 table.
(b) Determine an optimal mixed strategy for Casey.

Model solution

(a) Casey wants Ryan's payoff to be as small as possible. Column Y is smaller than column W in every row, so W is dominated. Column X is smaller than column Z in every row, so Z is dominated.

	X	Y
A	0	2
B	2	-3
C	4	-4
D	-1	5

Delete W and Z, giving the reduced table shown. **(2)**

(b) Let Casey choose X with probability p, and Y with probability $1-p$. Ryan's expected payoff for each possible row is then a linear function of p. Casey chooses p to make Ryan's greatest available payoff as small as possible.

A	$0p + 2(1-p) = 2 - 2p$
B	$2p - 3(1-p) = 5p - 3$
C	$4p - 4(1-p) = 8p - 4$
D	$-p + 5(1-p) = 5 - 6p$

The upper boundary is minimised where the two active lines, for C and D, meet.

$$8p - 4 = 5 - 6p$$

$$14p = 9, \qquad p = \frac{9}{14}, \qquad 1 - p = \frac{5}{14}.$$

Casey should choose X with probability $\frac{9}{14}$ and Y with probability $\frac{5}{14}$, choosing randomly between these cards. **(4)**

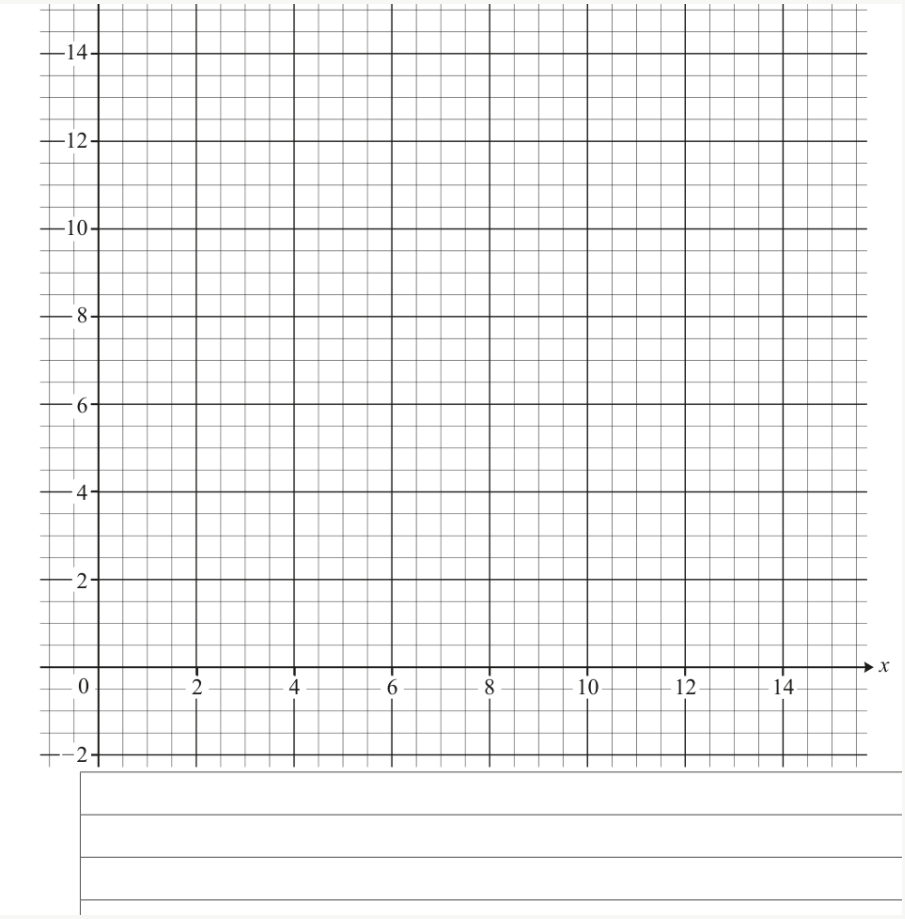
Question 7

Question 7

A linear programming problem is

$$\begin{aligned} &\text{Maximise } P = 4x + y \\ &\text{subject to } 3x - y \leq 30, \\ &\qquad x + y \leq 15, \\ &\qquad x - 3y \leq 6, \\ &\qquad x \geq 0,\ y \geq 0. \end{aligned}$$

(a) Use a graphical method to find the optimal value of P, and the corresponding values of x and y.



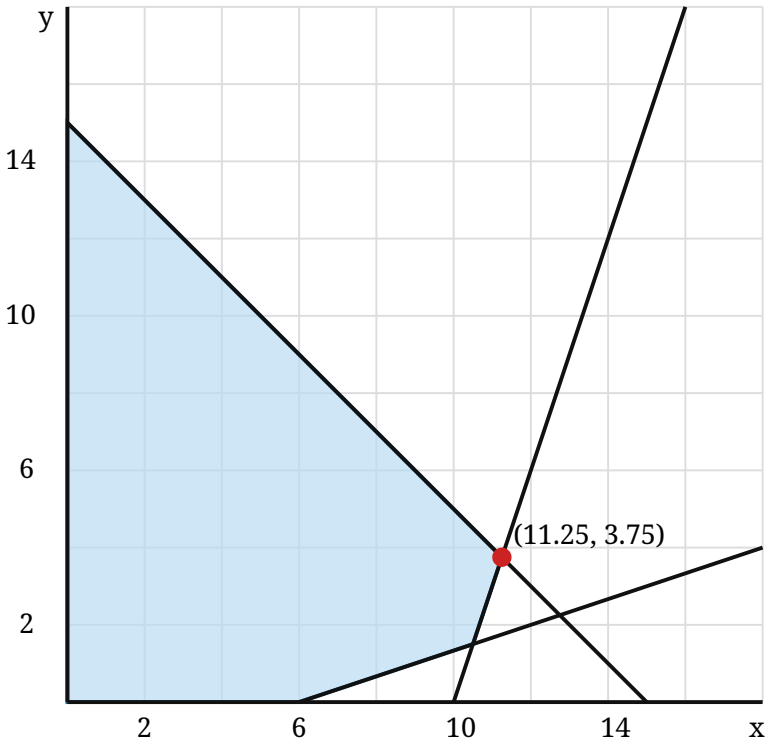
An additional constraint is introduced.
This constraint means that the value of y must be at least k times the value of x, where k is a positive constant.

(b) **(i)** Determine the set of values of k for which the optimal value of P found in part (a) is unchanged.

(ii) Determine, in terms of k, the values of x, y and P in the cases when the optimal solution is different from that found in part (a).

Model solution

(a) The boundary lines are $y = 3x - 30$, $y = 15 - x$, and $y = \frac{1}{3}x - 2$. The inequalities select the region above $y = 3x - 30$ and $y = \frac{1}{3}x - 2$, and below $y = 15 - x$, in the first quadrant. A linear objective has its maximum at a vertex of this feasible region.



x	y	$P = 4x + y$
0	0	0
6	0	24
10.5	1.5	43.5
11.25	3.75	48.75
0	15	15

The largest value occurs where $x + y = 15$ meets $x - 3y = 6$.

$P_{\max} = 48.75$, *at* $x = 11.25$ *and* $y = 3.75$. **(6)**

(b)(i) The new condition is $y \geq kx$. The old optimum remains feasible precisely when its coordinates satisfy this inequality.

$$k \leq \frac{y}{x} = \frac{3.75}{11.25} = \frac{1}{3}.$$

Since k is positive, the set is $0 < k \leq \frac{1}{3}$. **(2)**

(b)(ii) For $k > \frac{1}{3}$, the previous optimum is excluded. The new optimal vertex lies where the new boundary $y = kx$ meets $x + y = 15$.

$$x + kx = 15$$

$$x = \frac{15}{1+k}, \qquad y = \frac{15k}{1+k}.$$

$$P = 4x + y = 4\left(\frac{15}{1+k}\right) + \frac{15k}{1+k} = \frac{15(4+k)}{1+k}.$$

For $k > \frac{1}{3}$, $x = \frac{15}{1+k}$, $y = \frac{15k}{1+k}$, and $P = \frac{15(4+k)}{1+k}$. ***(4)***

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