Maths with Matt

Physics and maths resources

OCR AS Level Further Mathematics A

H235/Y534 - Discrete Mathematics

Wednesday 20 October 2021

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the OCR Wednesday 20 October 2021 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

A set consists of five distinct non-integer values, A, B, C, D and E. The set is partitioned into non-empty subsets and there are at least two subsets in each partition.
(a) Show that there are 15 different partitions into two subsets. [2]
(b) Show that there are 25 different partitions into three subsets. [4]
(c) Calculate the total number of different partitions. [2]
The numbers $12, 24, 36, 48, 60, 72, 84$ and 96 are marked on a number line. The number line is then cut into pieces by making cuts at A, B, C, D and E, where $0 < A < B < C < D < E < 100$.
(d) Explain why there must be at least one piece with two or more of the numbers $12, 24, 36, 48, 60, 72, 84$ and 96. [2]

Model solution

(a) With two subsets, the smaller subset has either one member or two members. This avoids counting the same partition twice by choosing both a subset and its complement.

$\binom{5}{1} = 5$ *partitions of sizes* 1 *and* 4, $\binom{5}{2} = 10$ *partitions of sizes* 2 *and* 3.

There are $5 + 10 = 15$ *partitions.* **(2)**

(b) The possible subset-size patterns are $1, 1, 3$ *and* $1, 2, 2$. *Count each pattern separately, allowing for the two equal-sized subsets.*

sizes	*number of partitions*
$1, 1, 3$	$\binom{5}{3} = 10$
$1, 2, 2$	$5\left(\frac{\binom{4}{2}}{2}\right) = 5(3) = 15$

$10 + 15 = 25$ *partitions into three subsets.* **(4)**

(c) There are also partitions into four and five subsets. Their subset-size patterns are $1, 1, 1, 2$ *and* $1, 1, 1, 1, 1$.

$\binom{5}{2} = 10, \qquad 1.$

Total $= 15 + 25 + 10 + 1 = 51.$ **(2)**

(d) Five cuts make six pieces. There are eight marked numbers to place in only six pieces, so the pigeonhole principle applies.

At least one of the six pieces contains at least two of the eight numbers. **(2)**

Question 2

Question 2

Seven items need to be packed into bins. Each bin has capacity 30 kg. The sizes of the items, in kg, in the order that they are received, are

$$12 \quad 23 \quad 15 \quad 18 \quad 8 \quad 7 \quad 5.$$

(a) Find the packing that results using each of these algorithms.
(i) The next-fit method [2]
(ii) The first-fit method [2]
(iii) The first-fit decreasing method [2]
(b) A student claims that all three methods from part (a) can be used for both 'online' and 'offline' lists. Explain why the student is wrong. [2]
The bins of capacity 30 kg are replaced with bins of capacity M kg, where M is an integer. The item of size 23 kg can be split into two items, of sizes x kg and $(23 - x)$ kg, where x may be any integer value you choose from 1 to 11. No other item can be split.
(c) Determine the smallest value of M for which four bins are needed to pack these eight items. Explain your reasoning carefully. [3]

Model solution

(a)(i) Next-fit considers only the bin most recently opened. Start a new bin whenever the next item will not fit in that bin.

bin	*contents (kg)*	*total (kg)*
1	12	12
2	23	23
3	15	15
4	18, 8	26
5	7, 5	12

The next-fit packing is $(12), (23), (15), (18, 8), (7, 5)$. *(2)*

(a)(ii) First-fit tests bins from bin 1 onwards, putting an item into the first bin with enough space.

bin	*contents (kg)*	*total (kg)*
1	12, 15	27
2	23, 7	30
3	18, 8	26
4	5	5

The first-fit packing is $(12, 15), (23, 7), (18, 8), (5)$. *(2)*

(a)(iii) First-fit decreasing is first-fit after sorting the complete list from largest to smallest.

$$23, \ 18, \ 15, \ 12, \ 8, \ 7, \ 5$$

bin	*contents (kg)*	*total (kg)*
1	23, 7	30
2	18, 12	30
3	15, 8, 5	28

The first-fit decreasing packing is $(23, 7), (18, 12), (15, 8, 5)$. *(2)*

(b) An online list reveals items one at a time; an offline list is known in full before packing begins.

Next-fit and first-fit can use either type because they use the arrival order. First-fit decreasing needs the whole list in order to sort it first, so it cannot be used for an online list. **(2)**

(c) The total size is $12+23+15+18+8+7+5=88$ *kg, so four bins require* $M \geq 88/4 = 22$*. Test the lower bound before constructing a packing. If* $M = 22$*, the item* 18 *can only be paired with a split piece of size at most* 4*. Thus the* 23 *must split as* $4+19$*; but then the* 19 *needs a companion of size at most* 3*, which is unavailable. Hence* $M = 22$ *is impossible.*

bin capacity 23	contents
1	$23-x,\ x$
2	$18,\ 5$
3	$15,\ 8$
4	$12,\ 7$

The first bin has total 23*, whatever permitted value of* x *is chosen, and the other bins have totals* $23, 23, 19$*.*

The smallest possible value is $M = 23$ *kg.* **(3)**

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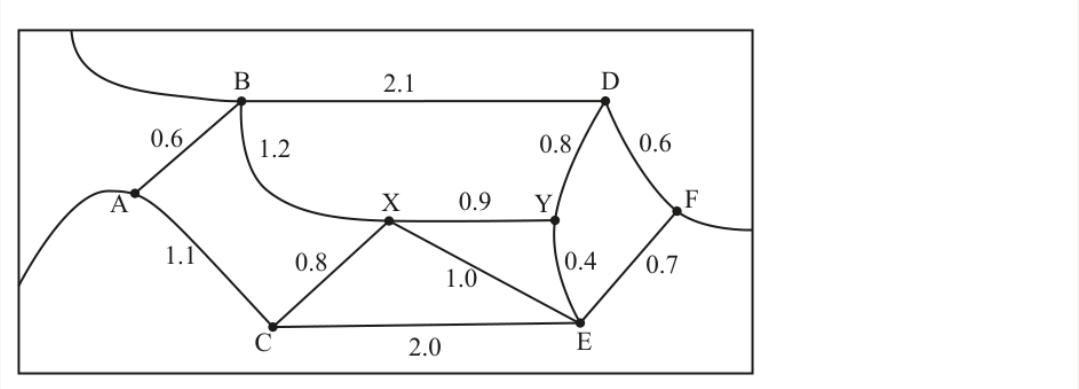
Question 3

Question 3

The diagram shows a simplified map of the main streets in a small town.



Some of the junctions have traffic lights; these junctions are labelled A to F. There are no traffic lights at junctions X and Y. The numbers show distances, in km, between junctions.
Alex needs to check that the traffic lights at junctions A to F are working correctly.
(a) Find a route from A to E that has length 2.8 km. [1]
Alex has started to construct a table of shortest distances between junctions A to F.

	A	B	C	D	E	F
A	A					
B		B				
C		1.7	C			
D		2.1	2.5	D		
E		2.2	1.8	1.2	E	
F						F

For example, the shortest route from C to B has length 1.7 km, the shortest route from C to D has length 2.5 km and the shortest route from C to E has length 1.8 km.
(b) Complete the copy of the table in the Printed Answer Booklet. [4]
(c) Use your table from part (b) to construct a minimum spanning tree for the complete graph on the six vertices A to F.
• Write down the total length of the minimum spanning tree.
• List which arcs of the original network correspond to the arcs used in your minimum spanning tree. [3]
Beth starts from junction B and travels through every junction, including X and Y. Her route has length 5.1 km.
(d) Write down the junctions in the order that Beth visited them. Do not draw on your answer from part (c). [2]

Model solution

(a) Add arc lengths along a route from A to E.

$A - B - X - E$, *with length* $0.6 + 1.2 + 1.0 = 2.8$ *km.* *(1)*

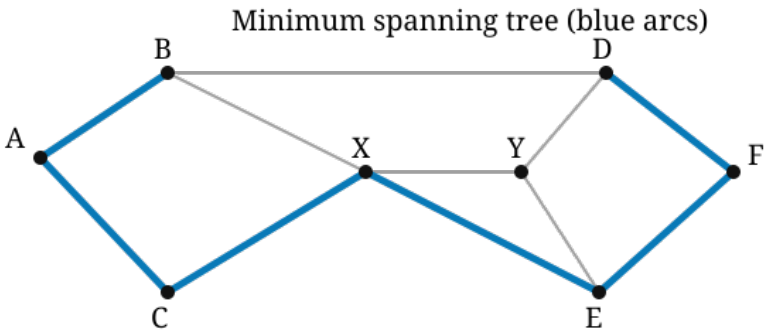
(b) For each pair of traffic-light junctions, compare possible routes through X and Y as well as

direct arcs. The completed symmetric shortest-distance table is:

	A	B	C	D	E	F
A	0	0.6	1.1	2.7	2.8	3.3
B	0.6	0	1.7	2.1	2.2	2.7
C	1.1	1.7	0	2.5	1.8	2.5
D	2.7	2.1	2.5	0	1.2	0.6
E	2.8	2.2	1.8	1.2	0	0.7
F	3.3	2.7	2.5	0.6	0.7	0

Completed table as shown. **(4)**

(c) Apply Kruskal's method to the complete graph: choose the shortest available distances without making a cycle. The complete-graph arc CE is realised in the original network by $C - X - E$.



$$AB = 0.6, \quad DF = 0.6, \quad EF = 0.7, \quad AC = 1.1, \quad CE = 1.8.$$

Total length $= 4.8$ km; original-network arcs: AB, AC, CX, XE, EF, FD. **(3)**

(d) Check the stated length by adding consecutive arcs in a route which starts at B and includes X and Y.

$$BA + AC + CX + XY + YE + EF + FD = 0.6 + 1.1 + 0.8 + 0.9 + 0.4 + 0.7 + 0.6 = 5.1.$$

$B - A - C - X - Y - E - F - D$. **(2)**

Question 4

Question 4

Li and Mia play a game in which they simultaneously play one of the strategies X, Y and Z. The tables show the points won by each player for each combination of strategies. A negative entry means that the player loses that number of points.

Points won by Li

Li\Mia	X	Y	Z
X	5	-6	0
Y	-2	3	4
Z	-1	4	8

Points won by Mia

Li\Mia	X	Y	Z
X	4		
Y	11	5	
Z	10	5	1

The game can be converted into a zero-sum game; this means that the total number of points won by Li and Mia is the same for each combination of strategies.
(a) (i) Complete the table in the Printed Answer Booklet to show the points won by Mia. [1]
(ii) Convert the game into a zero-sum game, giving the pay-offs for Li. [1]
(b) Use dominance to reduce the pay-off matrix for the game to a 2×2 table. You do not need to explain the dominance. [2]
Mia knows that Li will choose his play-safe strategy.
(c) Determine which strategy Mia should choose to maximise her points. [3]

Model solution

(a)(i) The total at each known position is 9, so subtract Li's entry from 9 to complete Mia's table.

Points won by Mia

Li\Mia	X	Y	Z
X	4	15	9
Y	11	5	6
Z	10	5	1

Completed table as shown. *(1)*

(a)(ii) Subtract 4.5 from every one of Li's pay-offs. Since the original totals are 9, the new two players' pay-offs total zero.

Zero-sum pay-offs for Li

Li\Mia	X	Y	Z
X	0.5	-10.5	-4.5
Y	-6.5	-1.5	-0.5
Z	-5.5	-0.5	3.5

The zero-sum pay-off matrix for Li is shown above. *(1)*

(b) Li maximises pay-off, so row Z dominates row Y. Mia minimises Li's pay-off, so column Y dominates column Z.

Li\Mia	X	Y
X	0.5	-10.5
Z	-5.5	-0.5

The reduced matrix is shown above. *(2)*

(c) Li's play-safe choice maximises his row minimum.

Li's strategy	*row minimum*
X	-10.5
Z	-5.5

Li therefore chooses Z. In row Z, Mia maximises her own score by minimising Li's pay-off: -5.5 occurs in column X.

Mia should choose strategy X. **(3)**

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Question 5

Question 5

A linear programming problem is formulated as below.

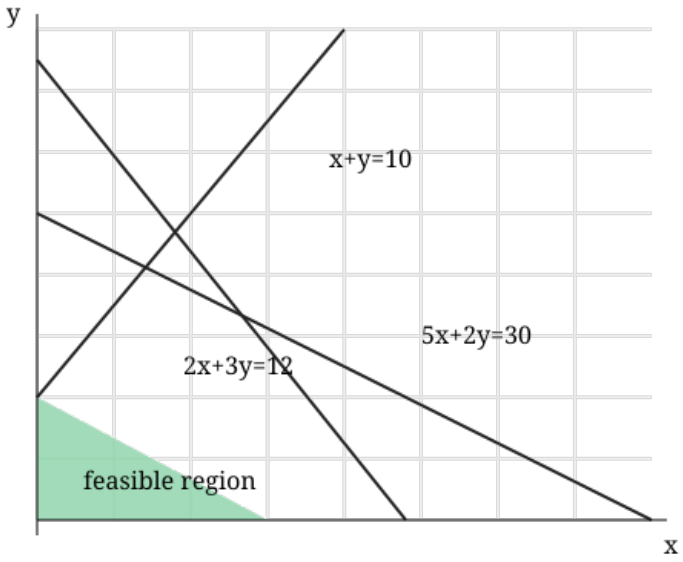
$$\text{Maximise} \quad P = 4x - y$$

$$\text{subject to} \quad 2x + 3y \leq 12, \qquad x + y \leq 10, \qquad 5x + 2y \leq 30, \qquad x, y \geq 0.$$

(a) (i) Identify the feasible region by representing the constraints graphically and shading the regions where the inequalities are not satisfied. [4]
(ii) Hence determine the maximum value of the objective. [2]
The constraint $x + y \leq 10$ is changed to $x + y \leq k$, the other constraints are unchanged.
(b) Determine, algebraically, the value of k for which the maximum value of P is 3. Do not draw on the graph from part (a) and do not use the spare grid. [3]
(c) Determine, algebraically, the other value of k for which there is a (non-optimal) vertex of the feasible region at which $P = 3$. Do not draw on the graph from part (a) and do not use the spare grid. [2]

Model solution

(a)(i) Draw each boundary as an equality and retain the side satisfying each inequality. The common unshaded region is the feasible region.



The active vertices are $(0,0)$, $(6,0)$ and $(0,4)$; the other two constraints do not further restrict this triangle.

Feasible region: the triangle with vertices $(0,0)$, $(6,0)$, $(0,4)$. **(4)**

(a)(ii) A linear objective reaches its maximum at a vertex, so evaluate $P = 4x - y$ at the feasible vertices.

(x, y)	$P = 4x - y$
$(0,0)$	0
$(6,0)$	24
$(0,4)$	-4

$\max P = 24$, *at* $(6, 0)$. *(2)*

(b) For the maximum to be 3*, the new boundary* $x + y = k$ *meets the boundary* $2x + 3y = 12$ *at the vertex on the profit line* $4x - y = 3$*. Solve these two equations first.*

$$4x - y = 3, \qquad 2x + 3y = 12.$$

$$y = 4x - 3, \qquad 2x + 3(4x - 3) = 12$$

$$14x = 21, \qquad x = \frac{3}{2}, \qquad y = 3.$$

$$k = x + y = \frac{3}{2} + 3 = \frac{9}{2}.$$

$k = \dfrac{9}{2} = 4.5$. *(3)*

(c) The other relevant boundary is $5x + 2y = 30$*. Its intersection with the profit line* $4x - y = 3$ *gives the required non-optimal vertex.*

$$y = 4x - 3, \qquad 5x + 2(4x - 3) = 30$$

$$13x = 36, \qquad x = \frac{36}{13}, \qquad y = \frac{105}{13}.$$

$$k = x + y = \frac{141}{13}.$$

$k = \dfrac{141}{13}$. *(2)*

Question 6

Question 6

Sarah is having some work done on her garden. The table below shows the activities involved, their durations and their immediate predecessors. These durations and immediate predecessors are known to be correct.

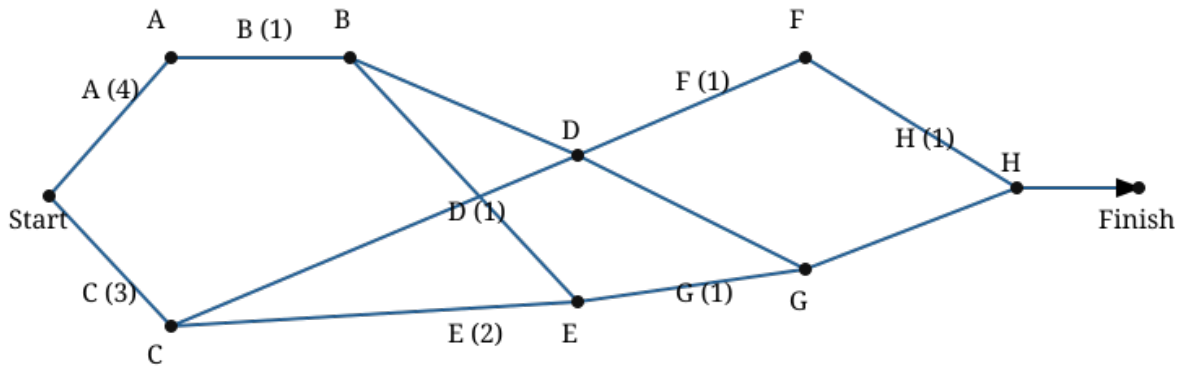
Activity	Immediate predecessors	Duration (hours)
A Clear site	–	4
B Mark out new design	A	1
C Buy materials, turf, plants and trees	–	3
D Lay paths	B, C	1
E Build patio	B, C	2
F Plant trees	D	1
G Lay turf	D, E	1
H Finish planting	F, G	1

(a) (i) Use a suitable model to determine the following.
- The minimum time in which the work can be completed
- The activities with zero float [6]

(ii) State one practical issue that could affect the minimum completion time in part (a)(i). [1]
Sarah needs the work to be completed as quickly as possible. There will be at least one activity happening at all times, but it may not always be possible to do all the activities that are needed at the same time.
(b) Determine the earliest and latest times at which building the patio (activity E) could start. [3]
There needs to be a 2-hour break after laying the paths (activity D). During this time other activities that do not depend on activity D can still take place.
(c) Describe how you would adapt your model to incorporate the 2-hour break. [1]

Model solution

(a)(i) An activity network is appropriate because immediate predecessors determine which activities may run in parallel. The unlabelled arcs are dummy activities, used to enforce the stated dependencies.



A forward pass gives earliest event times, and a backward pass from the finish gives latest event

times.

activity	A	B	C	D	E	F	G	H
earliest start	0	4	0	5	5	6	7	8
earliest finish	4	5	3	6	7	7	8	9
latest start	0	4	3	6	5	7	7	8
float	0	0	3	1	0	1	0	0

Minimum completion time $= 9$ hours. Activities with zero float are A, B, E, G, H. **(6)**

(a)(ii) The network assumes all required activities can be staffed whenever they are ready. In practice, limited workers or equipment can prevent this.

For example, insufficient workers may mean that activities which could run concurrently have to be scheduled one after another. **(1)**

(b) Activity E needs both B and C first, so its earliest start is when the later of these has finished: 5 hours. For the latest start, schedule all work that can precede E sequentially without creating an unnecessary delay: A, B, C, D, F take $4 + 1 + 3 + 1 + 1 = 10$ hours.

Activity E can start earliest at 5 hours and latest at 10 hours after the start. **(3)**

(c) The break delays every activity dependent on D, so build it into the activity network.

Replace the duration of D by 3 hours, equivalently add a 2-hour activity immediately after D. **(1)**

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