Maths with Matt

Physics and maths resources

OCR AS Level Further Mathematics A

H235/Y535 - Additional Pure Mathematics

Friday 17 June 2022

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the OCR Friday 17 June 2022 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

The points A, B and C have position vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -5 \\ 1 \\ 2 \end{pmatrix}$ respectively, relative to the origin O.

(a) Calculate, in its simplest exact form, the area of triangle OAB.

(b) Show that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$.

Model solution

(a) The area of a triangle formed by two position vectors is half the magnitude of their cross product.

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} -10 \\ 7 \\ 1 \end{pmatrix}, \qquad \text{area} = \frac{1}{2}|\mathbf{a} \times \mathbf{b}| = \frac{1}{2}\sqrt{(-10)^2 + 7^2 + 1^2} = \frac{\sqrt{150}}{2}.$$

The area is $\frac{5\sqrt{6}}{2}$ square units. **(3)**

(b) Evaluate the three vector triple products directly, then add their components.

$$\mathbf{b} \times \mathbf{c} = \begin{pmatrix} 7 \\ 1 \\ 17 \end{pmatrix}, \quad \mathbf{c} \times \mathbf{a} = \begin{pmatrix} 1 \\ 17 \\ -6 \end{pmatrix}, \quad \mathbf{a} \times \mathbf{b} = \begin{pmatrix} -10 \\ 7 \\ 1 \end{pmatrix}.$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \begin{pmatrix} 14 \\ 4 \\ -6 \end{pmatrix}, \quad \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) = \begin{pmatrix} -1 \\ 11 \\ 31 \end{pmatrix}, \quad \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \begin{pmatrix} -13 \\ -15 \\ -25 \end{pmatrix}.$$

$$\begin{pmatrix} 14 \\ 4 \\ -6 \end{pmatrix} + \begin{pmatrix} -1 \\ 11 \\ 31 \end{pmatrix} + \begin{pmatrix} -13 \\ -15 \\ -25 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence the stated identity is shown. **(3)**

Question 2

Question 2

The surface S has equation $z = x^3 + y^3 - 2x^2 - 5y^2 + 3xy$. It is given that S has two stationary points; one at the origin, O, and the other at the point A. Determine the coordinates of A.

Model solution

A stationary point on a surface has both first partial derivatives equal to zero.

$$\frac{\partial z}{\partial x} = 3x^2 - 4x + 3y = 0, qquad \frac{\partial z}{\partial y} = 3y^2 - 10y + 3x = 0.$$

$$y = \frac{4}{3}x - x^2.$$

Substitute this into the second equation to eliminate y.

$$x(x-1)(9x^2 - 15x + 31) = 0.$$

The quadratic has discriminant $225 - 4(9)(31) < 0$, so its roots are not real. The root $x = 0$ gives the known origin; therefore use $x = 1$.

$$y = \frac{4}{3} - 1 = \frac{1}{3}, qquad z = 1 + \frac{1}{27} - 2 - \frac{5}{9} + 1 = -\frac{14}{27}.$$

$$A = \left(1, \frac{1}{3}, -\frac{14}{27}\right). \quad \textit{(6)}$$

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Question 3

Question 3

The sequence $\{U_n\}$ is given by $U_1 = 0$, $U_2 = -1$ and $U_{n+2} = U_{n+1} + U_n + n - 1$ for $n \geq 1$.
(a) List the first seven terms of this sequence.
The Fibonacci sequence $\{F_n\}$ is given by $F_1 = 1$, $F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$.
(b) (i) By comparing the two sequences, give the relationship between U_n and F_n.
(ii) Show that the relationship found in part (b)(i) holds for all $n \geq 1$.

Model solution

(a) Apply the recurrence successively, using $n = 1, 2, \ldots$.

$$U_3 = -1 + 0 + 0 = -1, \quad U_4 = -1 - 1 + 1 = -1, \quad U_5 = -1 - 1 + 2 = 0,$$

$$U_6 = 0 - 1 + 3 = 2, \qquad U_7 = 2 + 0 + 4 = 6.$$

$0, -1, -1, -1, 0, 2, 6.$ **(2)**

(b)(i) Adding the term number to U_n gives the Fibonacci values.

$F_n = U_n + n.$ **(2)**

(b)(ii) Substitute $F_r = U_r + r$ into the Fibonacci recurrence; this proves that the stated relationship generates exactly the recurrence for U_n.

$$F_{n+2} = F_{n+1} + F_n$$

$$U_{n+2} + n + 2 = (U_{n+1} + n + 1) + (U_n + n),$$

$$U_{n+2} = U_{n+1} + U_n + n - 1.$$

This is the given recurrence. Also $U_1 + 1 = 1 = F_1$ and $U_2 + 2 = 1 = F_2$, so the initial values agree. Thus the relationship holds for every $n \geq 1$.

$F_n = U_n + n$ **for all** $n \geq 1.$ **(3)**

Question 4

Question 4

Let $N = 10a + b$ and $M = a + 3b$, where a and b are integers such that $a \geq 1$ and $0 \leq b \leq 9$.
(a) Prove that $29 \mid N$ if and only if $29 \mid M$.
(b) Use an iterative method based on the result of part (a) to show that $899\,364\,472$ is a multiple of 29.

Model solution

(a) A linear combination is useful because it cancels b:

$$3N - M = 3(10a + b) - (a + 3b) = 29a.$$

If $29 \mid N$, then $29 \mid 3N$, and hence $29 \mid (3N - 29a) = M$. Conversely, if $29 \mid M$, then $29 \mid (M + 29a) = 3N$. Since $\gcd(3, 29) = 1$, Euclid's lemma gives $29 \mid N$.

Therefore $29 \mid N$ if and only if $29 \mid M$. **(5)**

(b) Repeatedly replace the final digit b of a number $10a + b$ by $3b$, preserving divisibility by 29.

$$899364472 \to 89936447 + 6 = 89936453 \to 8993645 + 9 = 8993654$$

$$\to 899365 + 12 = 899377 \to 89937 + 21 = 89958 \to 8995 + 24 = 9019$$

$$\to 901 + 27 = 928 \to 92 + 24 = 116 = 4 \times 29.$$

Since the final number is divisible by 29, $899\,364\,472$ is divisible by 29. **(3)**

Question 5

Question 5

A research student is using 3-D graph-plotting software to model a chain of volcanic islands in the Pacific Ocean. These islands appear above sea-level at regular intervals, (approximately) distributed along a straight line. Each island takes the form of a single peak; also, along the line of islands, the heights of these peaks decrease in size in an (approximately) regular fashion (see Fig. 1.1).

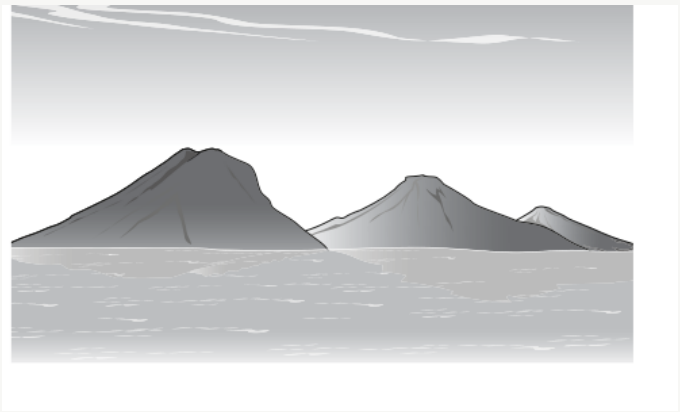


Fig. 1.1

The student's model uses the surface with equation $z = \sin x + \sin y$, a part of which is shown in Fig. 1.2 below. The surface of the sea is taken to be the plane $z = 0$.

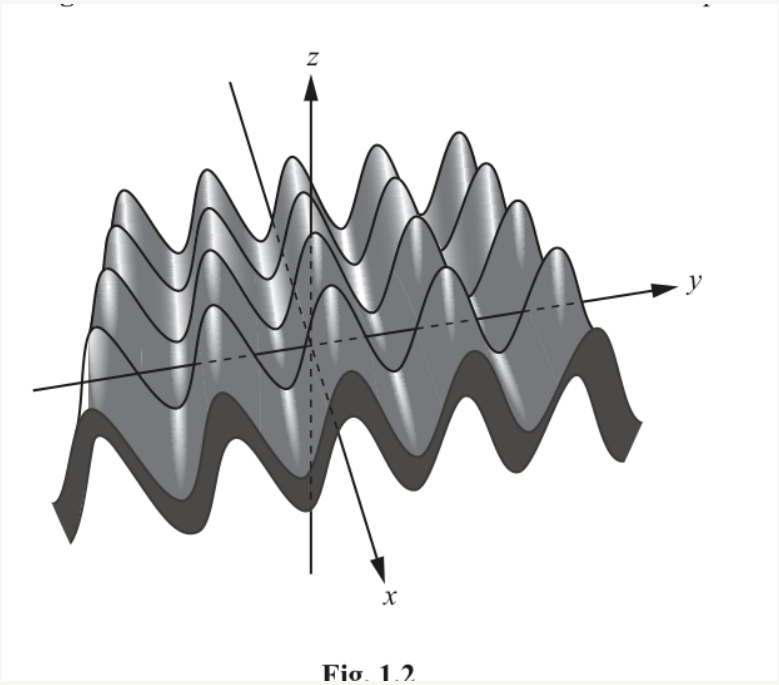


Fig. 1.2

(a) • Describe two problems with this model.
• Suggest revisions to this model so that each of these problems is addressed.
(b) Still using their original model, the student examines the contour $z = 2$ for their surface only to find that the software shows what appears to be an empty graph. Explain what has happened.

Model solution

(a) The model must reproduce a single line of islands whose peak heights decrease. In $\sin x + \sin y$*, periodicity in both variables creates repeated lines of peaks in two directions, and every maximum has the same height.*

$$\text{Restrict, for example, } -\frac{\pi}{2} \leq y \leq \frac{3\pi}{2}$$

to keep one line of peaks. Multiply by a damping factor such as e^{-ax}*, where* $a > 0$*, so that the peak heights decrease in the* x*-direction.*

The two defects are repeated rows of islands and equal peak heights; restricting one coordinate and adding damping addresses them. **(4)**

(b) Since each sine is at most 1*,* $\sin x + \sin y \leq 2$*. Equality requires* $\sin x = \sin y = 1$*, so the contour consists only of isolated maximum points, rather than a visible curve.*

The apparently empty plot is the set of isolated peak points where $z = 2$*.* **(2)**

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Question 6

Question 6

The sequence $\{u_n\}$ is such that $u_1 = 7$, $u_2 = 37$, $u_3 = 337$, $u_4 = 3337$, ...
(a) Write down a first-order recurrence system for $\{u_n\}$.
(b) By solving the recurrence system of part (a), show that $u_n = \frac{1}{3}(10^n + 11)$.
(c) Prove that $\{u_n\}$ contains infinitely many terms which are multiples of 37.

Model solution

(a) Each term appends a digit 3 before the final 7, so the successive increments are $30, 300, 3000, \ldots$.

$u_1 = 7$, $u_{n+1} = u_n + 3 \times 10^n$ *for* $n \geq 1$. *(2)*

(b) Solve the recurrence as a constant complementary solution plus a particular solution proportional to 10^n. *Let* $u_n = A\,10^n + C$.

$$A10^{n+1} + C = A10^n + C + 3 \times 10^n$$

$$9A = 3, \qquad A = \frac{1}{3}.$$

$$u_n = \frac{1}{3}10^n + C, \qquad 7 = \frac{10}{3} + C, qquad C = \frac{11}{3}.$$

$u_n = \frac{1}{3}(10^n + 11)$. *(5)*

(c) Work modulo 37. *The first few terms give* $u_1, u_2, u_3, u_4, u_5 \equiv 7, 0, 4, 7, 0 \pmod{37}$, *suggesting that every third term after* u_2 *is divisible by* 37. *Prove the three-step pattern by induction.*
The base case is $u_2 = 37$, *a multiple of* 37. *Assume* $37 \mid u_k$. *Applying the recurrence three times gives*

$$u_{k+3} = u_k + 3\left(10^k + 10^{k+1} + 10^{k+2}\right) = u_k + 333 \times 10^k.$$

$$333 = 9 \times 37,$$

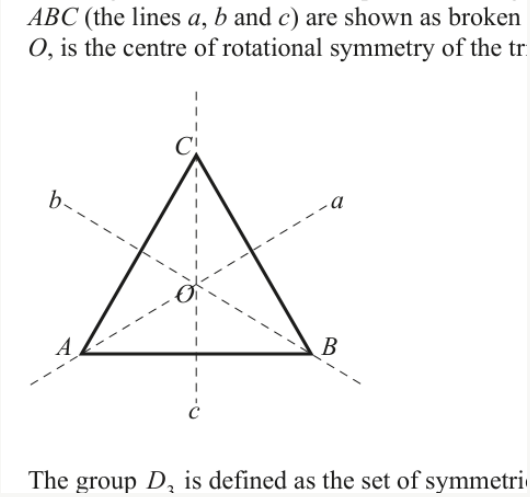
so $37 \mid u_k$ *implies* $37 \mid u_{k+3}$. *By induction,* u_{2+3r} *is a multiple of* 37 *for every integer* $r \geq 0$. *There are infinitely many such indices.*

$\{u_n\}$ *contains infinitely many multiples of* 37. *(7)*

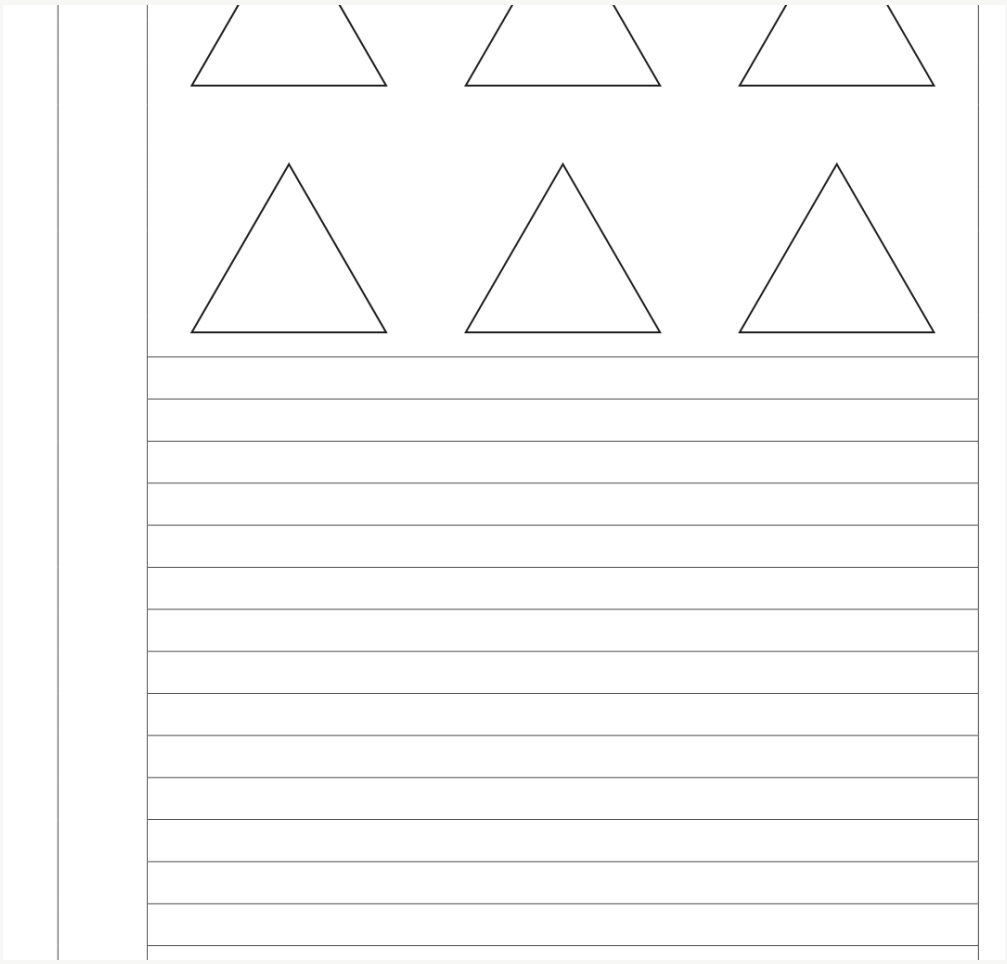
Question 7

Question 7

The diagram below shows an equilateral triangle ABC. The three lines of reflection symmetry of ABC, the lines a, b and c, are shown as broken lines. The point of intersection of these three lines, O, is the centre of rotational symmetry of the triangle.



The group D_3 is defined as the set of symmetries of ABC under the composition of the following transformations.
i: the identity transformation
a: reflection in line a
b: reflection in line b
c: reflection in line c
p: an anticlockwise rotation about O through $120°$
q: a clockwise rotation about O through $120°$ Note that the lines a, b and c are unaffected by the transformations and remain fixed.
(a) On the diagrams provided in the Printed Answer Booklet, show each of the six elements of D_3 obtained when the above transformations are applied to triangle ABC.

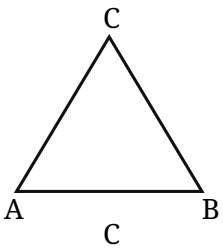


(b) Complete the Cayley table given in the Printed Answer Booklet.
(c) List all the proper subgroups of D_3.
(d) State, with justification, whether D_3 is (i) cyclic, (ii) abelian.
(e) The group H, also of order 6, is the set of rotational symmetries of the regular hexagon. Describe two structural differences between D_3 and H.

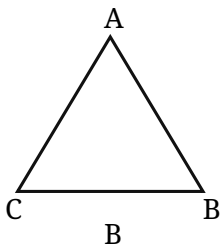
Model solution

(a) Keep the symmetry axes fixed and place each original vertex label at its image. Listing the labels at positions A, B, C, respectively, gives:

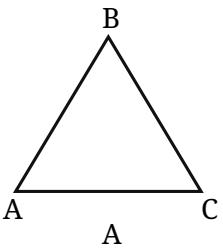
i



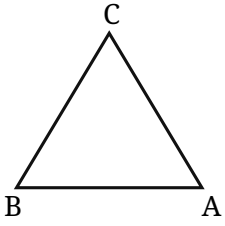
a



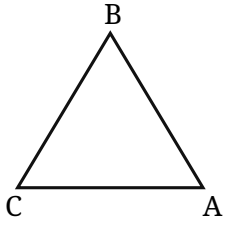
b



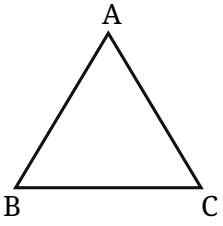
c



p



q



element	i	a	b	c	p	q
labels at (A,B,C)	(A,B,C)	(C,B,A)	(A,C,B)	(B,A,C)	(C,A,B)	(B,C,A)

These are the six completed triangle images. **(3)**

(b) Use the convention in the booklet: the row transformation is followed by the column transformation. For example, $ab = p$ *whereas* $ba = q$.

	i	a	b	c	p	q
i	i	a	b	c	p	q
a	a	i	p	q	b	c
b	b	q	i	p	c	a
c	c	p	q	i	a	b
p	p	c	a	b	q	i
q	q	b	c	a	i	p

Completed Cayley table as shown. **(3)**

(c) A proper subgroup must contain the identity and be closed. Each reflection generates a subgroup of order 2*, while either non-identity rotation generates the rotation subgroup of order* 3.

$\{i,a\},\ \{i,b\},\ \{i,c\},\ \{i,p,q\}$. **(3)**

(d)(i) A cyclic group of order 6 *would have an element of order* 6*. Here reflections have order* 2*, rotations* p,q *have order* 3*, and* i *has order* 1.

D_3 *is not cyclic.* **(1)**

(d)(ii) The Cayley table has unequal symmetric entries: $ab = p$*, but* $ba = q$*, and* $p \neq q$.

D_3 *is not abelian.* **(1)**

(e) The rotations of a regular hexagon form a cyclic group of order 6*, so they commute. This contrasts with the reflections and rotations in* D_3.

H *is cyclic and abelian;* D_3 *is neither.* **(2)**