Maths with Matt

Physics and maths resources

OCR AS Level Further Mathematics A

H235/Y535 - Additional Pure Mathematics

Thursday 21 October 2021

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the OCR Thursday 21 October 2021 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

Question wording is reproduced from the user-provided 2021 paper for study/reference. OCR retains copyright in the original examination material.

Contents

Question 1 2

Question 2 3

Question 3 5

Question 4 6

Question 5 7

Question 6 9

Question 7 11

Question 8 12

www.mathswithmatt.co.uk

Question 1

Question 1

The points A, B and C have position vectors

$$\mathbf{a} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}, \qquad \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

respectively, relative to the origin O.
(a)(i) Calculate $\mathbf{a} \times \mathbf{b}$, giving your answer as a multiple of $\mathbf{c}$. [1]
(a)(ii) Explain, geometrically, why $\mathbf{a} \times \mathbf{b}$ must be a multiple of $\mathbf{c}$. [1]
(b) Use a vector product method to calculate the area of triangle ABC. [3]

Model solution

(a)(i) Use the unit-vector cross-product rule because the two vectors lie along the x- and y-axes.

$$\mathbf{a} \times \mathbf{b} = (3\mathbf{i}) \times (4\mathbf{j}) = 12\mathbf{k} = 12\mathbf{c}.$$

$\mathbf{a} \times \mathbf{b} = 12\mathbf{c}$. *(1)*

(a)(ii) A vector product is perpendicular to each of its two input vectors. Here $\mathbf{a}$ *and* $\mathbf{b}$ *are in the x- and y-directions, so their perpendicular direction is the z-direction, which is the direction of* $\mathbf{c}$.

$\mathbf{a} \times \mathbf{b}$ *is a multiple of* $\mathbf{c}$. *(1)*

(b) Two sides from the same vertex give the vector-product area formula. The product gives the parallelogram area, so halve it for a triangle.

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} -3 \\ 4 \\ 0 \end{pmatrix}, \qquad \overrightarrow{AC} = \mathbf{c} - \mathbf{a} = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}.$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 4 \\ 3 \\ 12 \end{pmatrix}, \qquad \left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \sqrt{4^2 + 3^2 + 12^2} = 13.$$

$$\text{Area of } ABC = \frac{1}{2}(13) = \frac{13}{2}.$$

Area $= \frac{13}{2}$ *square units.* *(3)*

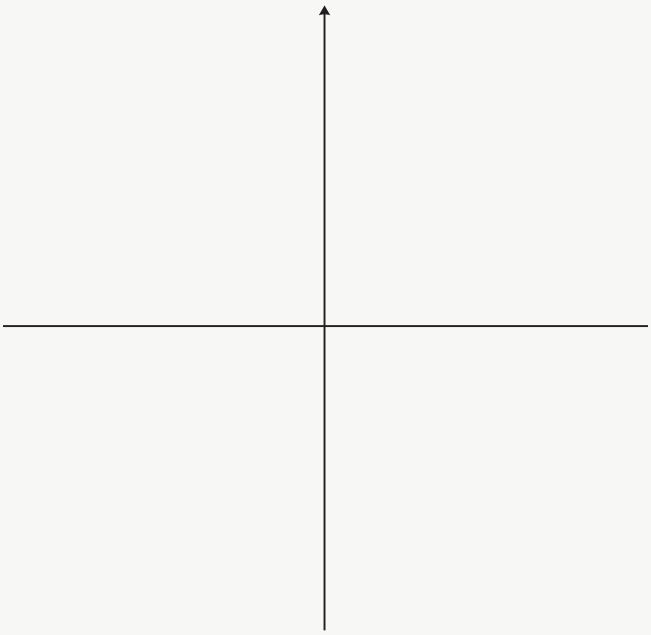
Question 2

Question 2

The surface S is given by

$$z = x^2 + 4xy$$

for $-6 \leq x \leq 6$ and $-2 \leq y \leq 2$.
(a)(i) Write down the equation of any one section of S which is parallel to the x-z plane. [1]
(a)(ii) Sketch the section of (a)(i) on the axes provided in the Printed Answer Booklet. [2]



(b) Write down the equation of any one contour of S which does not include the origin. [1]

Model solution

(a)(i) A section parallel to the x-z plane has a constant y-value. Choose the permitted value $y = 1$, then substitute it into the surface equation.

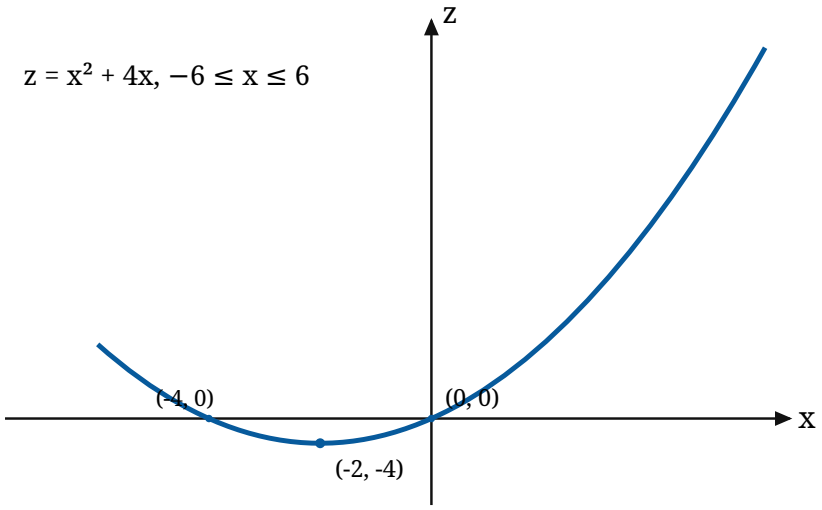
$$z = x^2 + 4x.$$

$z = x^2 + 4x$, **for** $-6 \leq x \leq 6$. ***(1)***

(a)(ii) Complete the square to locate the turning point accurately before sketching the upward-opening parabola.

$$z = x^2 + 4x = (x + 2)^2 - 4.$$

Thus the curve passes through $(0, 0)$ and has minimum $(-2, -4)$.



The required section is the shown upward-opening parabola. *(2)*

(b) A contour has constant height z. Taking $z = 1$, which is non-zero, ensures that this contour does not contain the origin.

$$1 = x^2 + 4xy.$$

$1 = x^2 + 4xy.$ *(1)*

www.mathswithmatt.co.uk

Question 3

Question 3

For positive integers n, the sequence of Fibonacci numbers, $\{F_n\}$, starts with the terms

$$F_1 = 1, \qquad F_2 = 1, \qquad F_3 = 2, \ldots$$

and is given by the recurrence relation

$$F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 3.$$

(a) Show that $F_{3k+3} = 2F_{3k+1} + F_{3k}$, where k is a positive integer. [2]
(b) Prove by induction that F_{3n} is even for all positive integers n. [4]

Model solution

(a) Apply the recurrence twice, so that the index $3k+3$ is reduced to terms involving $3k$.

$$\begin{aligned} F_{3k+3} &= F_{3k+2} + F_{3k+1} \\ &= (F_{3k+1} + F_{3k}) + F_{3k+1} \\ &= 2F_{3k+1} + F_{3k}. \end{aligned}$$

$F_{3k+3} = 2F_{3k+1} + F_{3k}$, as required. **(2)**

(b) Induction is appropriate because the statement is indexed by every positive integer n. Part (a) supplies the link from one multiple of three to the next.
Base case: when $n = 1$, $F_{3n} = F_3 = 2$, which is even.
Inductive hypothesis: assume that F_{3k} is even for some positive integer k.
For the next case, part (a) gives

$$F_{3(k+1)} = F_{3k+3} = 2F_{3k+1} + F_{3k}.$$

The term $2F_{3k+1}$ is even, and F_{3k} is even by the hypothesis. Their sum is therefore even. Hence $F_{3(k+1)}$ is even.

Since the base case is true and the result for k implies the result for $k+1$, F_{3n} is even for every positive integer n. **(4)**

Question 4

Question 4

(a) Let $a = 1071$ and $b = 67$.
(i) Find the unique integers q and r such that $a = bq + r$, where $q \geq 0$ and $0 \leq r < b$. [1]
(ii) Hence express the answer to (a)(i) in the form of a linear congruence modulo b. [1]
(b) Use the fact that

$$358 \times 715 - 239 \times 1071 = 1$$

to prove that 715 and 1071 are co-prime. [4]

Model solution

(a)(i) Use the division algorithm, choosing the quotient so that the remainder is between 0 and 66.

$$1071 = 67(15) + 66.$$

$q = 15$ *and* $r = 66$. ***(1)***

(a)(ii) The division equation says that 1071 differs from 66 by a multiple of 67.

$1071 \equiv 66 \pmod{67}$. ***(1)***

(b) Any common divisor must divide every integer linear combination of the two numbers. The given identity is exactly such a combination.
Let $h = \text{hcf}(715, 1071)$. Then $h \mid 715$ and $h \mid 1071$, so

$$h \mid \big(358(715) - 239(1071)\big).$$

Using the stated identity, this becomes

$$h \mid 1.$$

The only positive divisor of 1 is 1, so $h = 1$.

$\text{hcf}(715, 1071) = 1$*; therefore 715 and 1071 are co-prime.* ***(4)***

Question 5

Question 5

A trading company deals in two goods. The formula used to estimate z, the total weekly cost to the company of trading the two goods, in tens of thousands of pounds, is

$$z = 0.9x + \frac{0.096y}{x} - x^2y^2,$$

where x and y are the masses, in thousands of tonnes, of the two goods.
You are given that $x > 0$ and $y > 0$.
(a) In the first week of trading, it was found that the values of x and y corresponded to the stationary value of z. Determine the total cost to the company for this week. [7]
(b) For the second week, the company intends to make a small change in either x or y in order to reduce the total weekly cost. Determine whether the company should change x or y. (You are not expected to say by how much the company should reduce its costs.) [4]

Model solution

(a) At a stationary value both first partial derivatives are zero. Differentiate with respect to one variable at a time, treating the other as constant.

$$\frac{\partial z}{\partial x} = 0.9 - \frac{0.096y}{x^2} - 2xy^2, \qquad \frac{\partial z}{\partial y} = \frac{0.096}{x} - 2x^2y.$$

Set both derivatives equal to zero. The second equation is simplest to rearrange for y:

$$\frac{0.096}{x} = 2x^2y \quad \Rightarrow \quad y = \frac{0.048}{x^3}.$$

Substitute this into the first stationary equation:

$$0 = 0.9 - \frac{0.096(0.048/x^3)}{x^2} - 2x\left(\frac{0.048}{x^3}\right)^2.$$

$$0 = 0.9 - \frac{0.004608}{x^5} - \frac{0.004608}{x^5} = 0.9 - \frac{0.009216}{x^5}.$$

$$x^5 = \frac{0.009216}{0.9} = 0.01024, \qquad x = 0.4.$$

$$y = \frac{0.048}{(0.4)^3} = 0.75.$$

Substituting these masses into the cost model gives

$$z = 0.9(0.4) + \frac{0.096(0.75)}{0.4} - (0.4)^2(0.75)^2 = 0.36 + 0.18 - 0.09 = 0.45.$$

Since z is measured in tens of thousands of pounds, 0.45 represents $0.45 \times £10000$.

The total weekly cost is $\$4500$. *(7)*

(b) Second partial derivatives show how z changes for a small one-variable change at the stationary point. A positive value means a local minimum in that direction; a negative value means a local maximum.

$$\frac{\partial^2 z}{\partial x^2} = \frac{0.192y}{x^3} - 2y^2, \qquad \frac{\partial^2 z}{\partial y^2} = -2x^2.$$

At $x = 0.4,\ y = 0.75$,

$$\frac{\partial^2 z}{\partial x^2} = \frac{0.192(0.75)}{(0.4)^3} - 2(0.75)^2 = 1.125 > 0,$$

$$\frac{\partial^2 z}{\partial y^2} = -2(0.4)^2 = -0.32 < 0.$$

Thus changing x *alone raises the cost near the stationary point, whereas changing* y *can lower it.*

The company should change y. **(4)**

www.mathswithmatt.co.uk

Question 6

Question 6

The set S consists of the following four complex numbers:

$$\sqrt{3}+\mathrm{i}, \qquad -\sqrt{3}-\mathrm{i}, \qquad 1-\mathrm{i}\sqrt{3}, \qquad -1+\mathrm{i}\sqrt{3}.$$

For $z_1, z_2 \in S$, the binary operation $\circ$ is defined by

$$z_1 \circ z_2 = \frac{1}{4}(1+\mathrm{i}\sqrt{3})z_1 z_2.$$

(a)(i) Complete the Cayley table for $(S, \circ)$ given in the Printed Answer Booklet. [3]

(a)(i)

	$\sqrt{3}+\mathrm{i}$	$-\sqrt{3}-\mathrm{i}$	$1-\mathrm{i}\sqrt{3}$	$-1+\mathrm{i}\sqrt{3}$
$\sqrt{3}+\mathrm{i}$		$1-\mathrm{i}\sqrt{3}$		
$-\sqrt{3}-\mathrm{i}$		$-1+\mathrm{i}\sqrt{3}$	$-\sqrt{3}-\mathrm{i}$	
$1-\mathrm{i}\sqrt{3}$	$\sqrt{3}+\mathrm{i}$			$-1+\mathrm{i}\sqrt{3}$
$-1+\mathrm{i}\sqrt{3}$			$-1+\mathrm{i}\sqrt{3}$	

(a)(ii) Verify that $(S, \circ)$ is a group. [4]
(a)(iii) State the order of each element of $(S, \circ)$. [1]
(b) Write down the only proper subgroup of $(S, \circ)$. [1]
(c)(i) Explain why $(S, \circ)$ is a cyclic group. [1]
(c)(ii) List all possible generators of $(S, \circ)$. [1]

Model solution

For compactness, write $p = \sqrt{3}+\mathrm{i}$, $q = -\sqrt{3}-\mathrm{i}$, $r = 1-\mathrm{i}\sqrt{3}$ and $s = -1+\mathrm{i}\sqrt{3}$. Substitute each row and column pair in the operation; for example, $p \circ p = \frac{1}{4}(1+\mathrm{i}\sqrt{3})(\sqrt{3}+\mathrm{i})^2 = s$.
(a)(i)

$\circ$	p	q	r	s
p	s	r	p	q
q	r	s	q	p
r	p	q	r	s
s	q	p	s	r

The completed Cayley table is shown. *(3)*

(a)(ii) The table contains only elements of S, establishing closure. Associativity follows because the operation is multiplication by a fixed complex constant followed by ordinary complex multiplication:

$$(z_1 \circ z_2) \circ z_3 = \frac{1}{16}(1+\mathrm{i}\sqrt{3})^2 z_1 z_2 z_3 = z_1 \circ (z_2 \circ z_3).$$

From the table, $r = 1-\mathrm{i}\sqrt{3}$ is the identity. The inverse information is also read directly from the

entries producing r:

$$r^{-1} = r, \qquad s^{-1} = s, \qquad p^{-1} = q, \qquad q^{-1} = p.$$

$(S, \circ)$ is closed and associative, has identity $1 - \mathrm{i}\sqrt{3}$, and every element has an inverse; hence it is a group. **(4)**

(a)(iii) *The order is the first positive power under $\circ$ that gives the identity $r = 1 - \mathrm{i}\sqrt{3}$. The table shows the repeated products.*

$$\mathrm{ord}(r) = 1, \qquad \mathrm{ord}(s) = 2, \qquad \mathrm{ord}(p) = \mathrm{ord}(q) = 4.$$

z	r	s	p	q
$\mathrm{ord}(z)$	1	2	4	4

(1)

(b) *A proper non-trivial subgroup of a group of order four must have order two and include the identity.*

$\{1 - \mathrm{i}\sqrt{3},\ -1 + \mathrm{i}\sqrt{3}\}$. **(1)**

(c)(i) *A group of order four is cyclic when it has an element of order four.*

$(S, \circ)$ is cyclic because it has elements of order 4. **(1)**

(c)(ii) *Exactly the elements of order four generate all four group elements.*

$\sqrt{3} + \mathrm{i}$ and $-\sqrt{3} - \mathrm{i}$. **(1)**

Question 7

Question 7

(a) Let

$$f(n) = 2^{4n+3} + 3^{3n+1}.$$

Use arithmetic modulo 11 to prove that $f(n) \equiv 0 \pmod{11}$ for all integers $n \geq 0$. [4]
(b) Use the standard test for divisibility by 11 to prove the following statements.
(i) $10^{33} + 1$ is divisible by 11. [2]
(ii) $10^{33} + 1$ is divisible by 121. [4]

Model solution

(a) Separate the powers into factors whose residues modulo 11 are easy to calculate.

$$\begin{aligned} f(n) &= 2^3(2^4)^n + 3(3^3)^n \\ &= 8(16)^n + 3(27)^n \\ &\equiv 8(5)^n + 3(5)^n \pmod{11} \\ &= 11(5)^n \equiv 0 \pmod{11}. \end{aligned}$$

$f(n) \equiv 0 \pmod{11}$ *for every integer* $n \geq 0$. *(4)*

(b)(i) The divisibility test for 11 says that a number is divisible by 11 when the difference between the sums of its alternating digits is divisible by 11.
$10^{33} + 1$ *is a 1, followed by 32 zeros, followed by a 1. The two non-zero digits lie in opposite alternating positions, so*

$$(\textit{sum in one set of positions}) - (\textit{sum in the other}) = 1 - 1 = 0.$$

$10^{33} + 1$ *is divisible by 11.* *(2)*

(b)(ii) Part (i) supplies one factor of 11. Divide by it, then apply the same digit test to show that the quotient supplies another factor.

$$\frac{10^{33}+1}{11} = \underbrace{9090\ldots90}_{15 \text{ copies of } 90}\,91.$$

For this quotient, the alternating digit-sum difference is

$$16(9) - 1 = 143 = 13 \times 11.$$

Therefore the quotient is divisible by 11, and the original number is divisible by 11×11.

$10^{33} + 1$ *is divisible by 121.* *(4)*

Question 8

Question 8

A sequence $\{u_n\}$ is defined by the recurrence system

$$u_1 = 1 \qquad \text{and} \qquad u_{n+1} = a - \frac{a^2}{2u_n} \quad \text{for } n \geq 1,$$

where a is a positive constant.
Determine with justification the behaviour of the sequence for all possible values of a. [7]

Model solution

Generate successive terms. This reveals whether the recurrence returns to its starting value and also identifies values of a that would make a denominator zero.

$$u_2 = a - \frac{a^2}{2} = \frac{a(2-a)}{2}.$$

$$\begin{aligned} u_3 &= a - \frac{a^2}{2u_2} \\ &= a - \frac{a^2}{a(2-a)} = a - \frac{a}{2-a} = \frac{a(1-a)}{2-a}. \end{aligned}$$

$$\begin{aligned} u_4 &= a - \frac{a^2}{2u_3} \\ &= a - \frac{a(2-a)}{2(1-a)} = \frac{a^2}{2(a-1)}. \end{aligned}$$

$$\begin{aligned} u_5 &= a - \frac{a^2}{2u_4} \\ &= a - \frac{a^2}{a^2/(a-1)} = a - (a-1) = 1. \end{aligned}$$

Thus, whenever the displayed terms are defined, the recurrence has period 4:

$$1, \quad \frac{a(2-a)}{2}, \quad \frac{a(1-a)}{2-a}, \quad \frac{a^2}{2(a-1)}, \quad 1, \ldots$$

If $a = 1$, then $u_3 = 0$, so the next term is undefined. If $a = 2$, then $u_2 = 0$, so the next term is undefined. These are the only positive excluded values.
To check that the period cannot be shorter, a return to $u_1 = 1$ at u_2, u_3, or u_4 would respectively require

$$\frac{a(2-a)}{2} = 1 \quad \Rightarrow \quad a^2 - 2a + 2 = 0.$$

$$\frac{a(1-a)}{2-a} = 1 \quad \Rightarrow \quad a^2 - 2a + 2 = 0, \qquad \frac{a^2}{2(a-1)} = 1 \quad \Rightarrow \quad a^2 - 2a + 2 = 0.$$

This quadratic has discriminant $4 - 8 = -4 < 0$, so it has no real, hence no positive, solution.

For $a > 0$, $a \neq 1, 2$, the sequence is periodic with period 4, with terms $1, \frac{a(2-a)}{2}, \frac{a(1-a)}{2-a}, \frac{a^2}{2(a-1)}$. For $a = 1$ or $a = 2$, the sequence becomes undefined.

(7)