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Physics and maths resources

OCR MEI A-Level Further Mathematics

H645/Y433 - Modelling with Algorithms

Tuesday 18 June 2019

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the supplied official OCR mark scheme. The original paper carries 60 marks. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Contents

Question 1 2

Question 2 4

Question 3 7

Question 4 9

Question 5 11

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Question 1

Question 1

Fig. 1 shows a network. The weights on the arcs are distances.

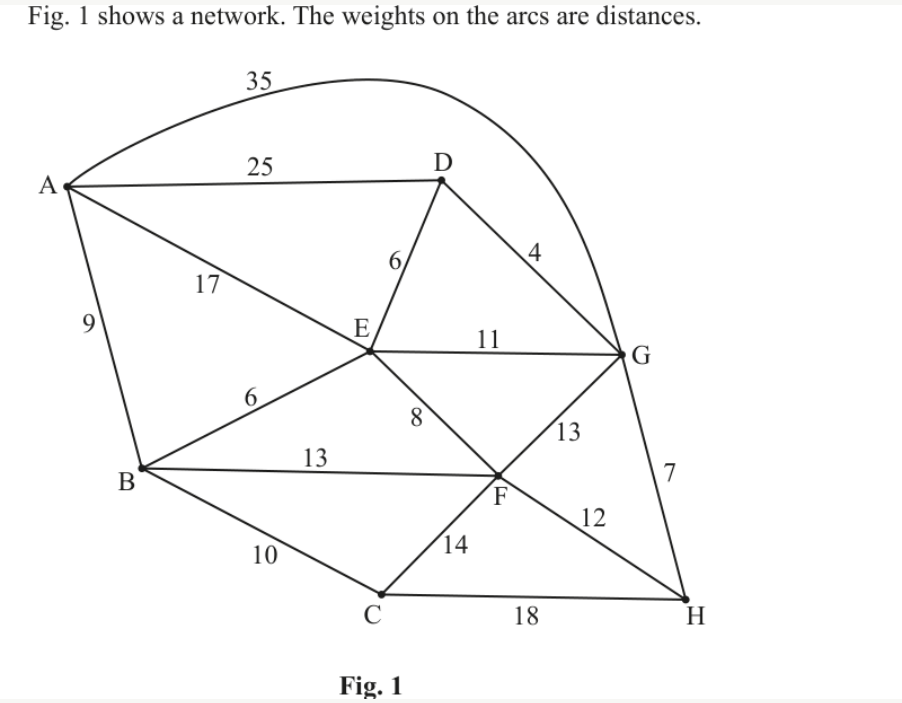


Fig. 1

(**a**) Apply Dijkstra's algorithm to find the shortest path from A to H. State the shortest path and its length. **(6)**

The weights in Fig. 1 are listed in descending order. (**b**) Use the first fit decreasing algorithm to pack these weights into bins of capacity 50. **(2)**

The number of comparisons is used as a measure of complexity. (**c**) A list of n weights is in descending order and is packed using first fit decreasing. Determine the worst-case complexity in terms of n. **(3)**

Model solution

(a) Dijkstra's algorithm is appropriate because all arc weights are positive. At each stage permanently label the unlabelled vertex with the smallest temporary distance from A.

vertex	*order made permanent*	*permanent label*	*successive working values*
A	1	0	0
B	2	9	9
E	3	15	$17, 15$
C	4	19	19
D	5	21	$25, 21$
F	6	22	$22, 23$
G	7	25	$35, 26, 25, 35$
H	8	32	$37, 34, 32$

Tracing back through the arcs which produced the final labels gives $H \leftarrow G \leftarrow D \leftarrow E \leftarrow B \leftarrow A$.

The shortest path is $A \to B \to E \to D \to G \to H$, with length $9 + 6 + 6 + 4 + 7 = 32$.

(6)

(b) Process the weights in descending order and place each one in the first already-open bin with enough remaining capacity.

bin	*contents*	*total*
1	$35, 14$	49
2	$25, 18, 7$	50
3	$17, 13, 13, 6$	49
4	$12, 11, 10, 9, 8$	50
5	$6, 4$	10

Five bins are used, as listed above.

(2)

(c) In the worst case, the second weight is compared with one bin, the third with two bins, and so on. Thus the last weight may need $n - 1$ comparisons.

$$1 + 2 + \cdots + (n - 1) = \frac{n(n-1)}{2}.$$

The worst-case complexity is $O(n^2)$.

(3)

Question 2

Question 2

The table gives the duration, in minutes, and immediate predecessors of each activity in a project.

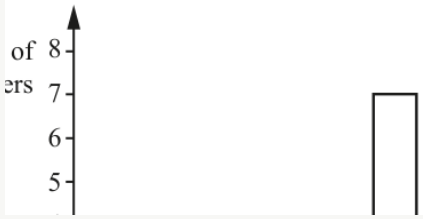
Activity	Duration	Immediate predecessors
A	25	–
B	20	–
C	35	–
D	15	A
E	25	A, B, C
F	25	A, B, C
G	30	C
H	20	E
I	25	E, F, G

(**a**) Draw an activity-on-arc network for the project. **(3)**
(**b**) Carry out a forward and backward pass, showing the early and late event time at each vertex, and list the critical activities. **(5)**
The following histogram shows the workers required when each activity begins at its earliest possible start time. Each activity requires at least one worker and, once started, must be completed without interruption.

'arry out a forward pass and a backward pass through the activity network, show
arly event time and the late event time at each vertex of your network.

ist the critical activities.

urce histogram in Fig. 2.2 shows the number of workers required when each
its earliest possible start time. It is given that each activity requires at least one
when an activity is started it must be completed without interruption.



(**c**) Complete the table giving the number of workers required for each activity. **(3)**
(**d**) Determine the shortest completion time when only four workers are available. **(2)**

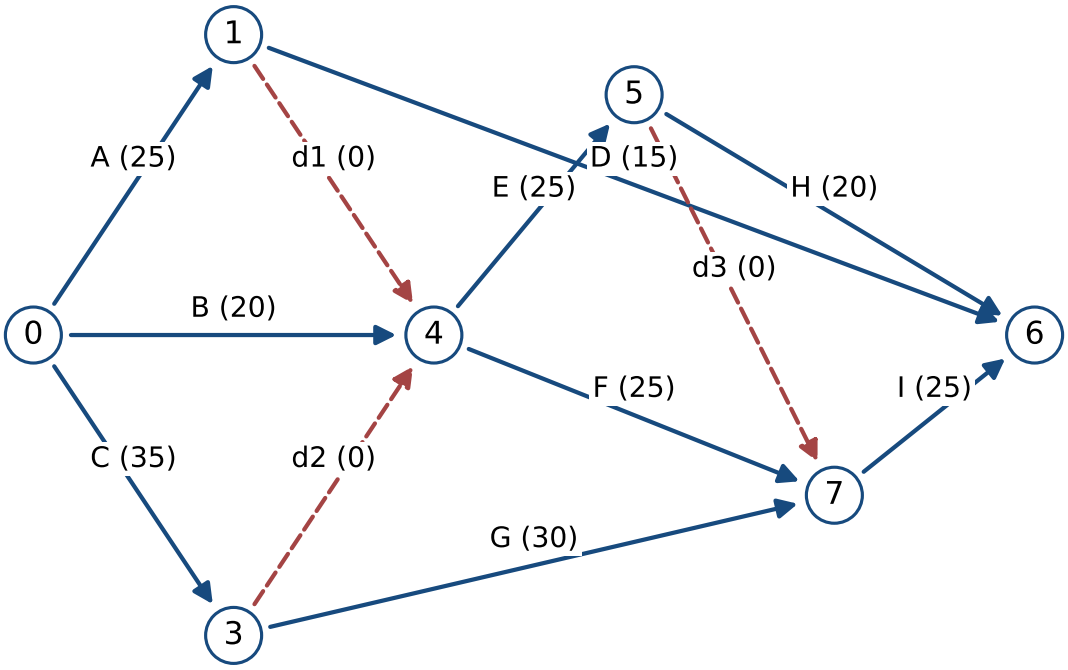
Model solution

(a) An activity-on-arc network must make an activity start only after all its stated predecessors have finished. One valid network is specified by the following arcs; d denotes a

dummy arc of duration 0.

arc	*activity (duration)*
$0 \to 1$	$A(25)$
$0 \to 4$	$B(20)$
$0 \to 3$	$C(35)$
$1 \to 6$	$D(15)$
$1 \to 4,\ 3 \to 4$	d, d
$4 \to 5$	$E(25)$
$4 \to 7$	$F(25)$
$3 \to 7$	$G(30)$
$5 \to 6$	$H(20)$
$5 \to 7$	d
$7 \to 6$	$I(25)$

One valid activity-on-arc network



The two dummy arcs into vertex 4 *ensure that both* E *and* F *wait for* A, B, C*, and the dummy from* 5 *to* 7 *makes* I *wait for* E *as well as* F, G. ***(3)***

(b) *For the forward pass, an event time is the greatest finish time of arcs entering it. For the backward pass, work from the project completion time, taking the least permissible time at each preceding event.*

vertex	0	1	3	4	5	7	6
early time	0	25	35	35	60	65	90
late time	0	40	35	40	65	65	90

An activity is critical when it has zero total float, found by comparing its earliest and latest possible start (or finish) times.

The critical activities are C, G, I, and the earliest completion time is 90 minutes.

(5)

(c) Compare the changes in the histogram with the known activity start and finish times. For example, at time 0, A, B, C require 4 workers, so their individual requirements must add to 4.

activity	A	B	C	D	E	F	G	H	I
workers	1	2	1	3	1	1	2	2	1

The worker requirements are as shown.

(3)

(d) The clash is from 35 to 40: D, E, F, G would need $3+1+1+2=7$ workers. Delay the non-critical activity D until after the other work which needs the workers.

activity	*start--finish*	*workers*
A	0--25	1
B	0--20	2
C	0--35	1
E, F, G	35--60, 35--60, 35--65	1, 1, 2
H	60--80	2
I	65--90	1
D	80--95	3

At no time does this schedule use more than four workers; D only depends on A, so its delay is allowed. To see why a shorter schedule is impossible, note that D needs three workers and cannot overlap either G or H, each of which needs two. If D precedes G, its earliest finish is $25+15=40$, so G then I cannot finish before $40+30+25=95$. If D follows G, it starts no earlier than 65; placing H before D gives earliest finish $60+20+15=95$, while putting H after D is even later. Thus 95 is an attained lower bound.

The shortest completion time is 95 minutes.

(2)

Question 3

Question 3

Fig. 3 is a flow network. Each arc weight is its capacity in litres per minute.

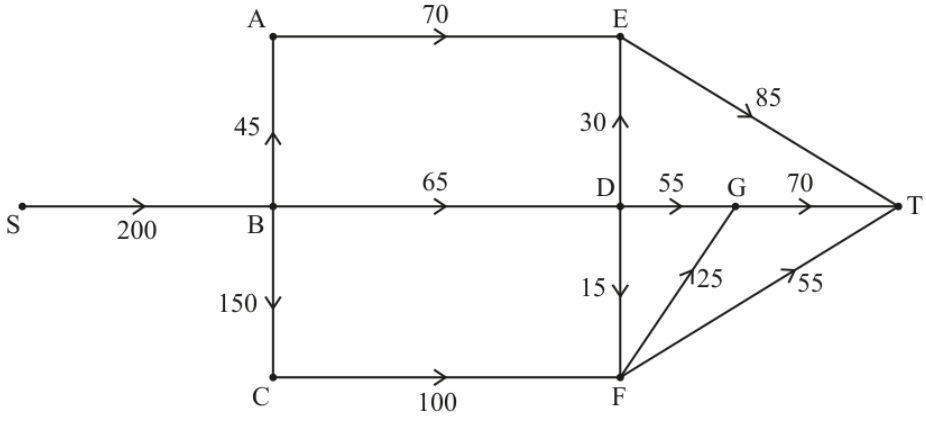


Fig. 3

(a) (i) Cut α partitions the vertices into $\{S, B, C\}$, $\{A, D, E, F, G, T\}$. Calculate its capacity. **(1)**

(a) (ii) Cut β partitions the vertices into $\{S, A, B, C, E, F\}$, $\{D, G, T\}$. Calculate its capacity. **(1)**

(b) Using only these two capacities, state what can be deduced about the maximum possible flow. **(1)**

(c) Show how a flow of 190 litres per minute can be achieved. **(2)**

(d) Use a suitable cut to prove this is the maximum possible flow. **(2)**

An extra pipe from S to A, of capacity x litres per minute, is opened. **(e)** Find the maximum possible flow, in terms of x where necessary. **(3)**

Model solution

(a) The capacity of a directed cut is found by adding only arcs directed from the first set to the second set.

$$\text{cap}(\alpha) = 45 + 65 + 100 = 210, \qquad \text{cap}(\beta) = 65 + 25 + 55 + 85 = 230.$$

$\text{cap}(\alpha) = 210$ *litres per minute;* $\text{cap}(\beta) = 230$ *litres per minute.*

(2)

(b) Every flow is at most the capacity of every cut, so use the smaller of the two cut capacities.

The maximum flow is at most 210 *litres per minute.*

(1)

(c) Give flows on all arcs so that flow in equals flow out at every internal vertex and no capacity is exceeded. One such feasible flow is:

arc	SB	BA	BC	AE	BD	CF	DE	DF	FG	DG	ET	FT	GT
flow	190	45	80	45	65	80	20	0	25	45	65	55	70

For example, at F, $80 + 0 = 80$ enters and $25 + 55 = 80$ leaves; all other internal vertices balance similarly.

A feasible flow of 190 litres per minute is achieved.

(2)

(d) Use the cut $\{S, B, C, F\}$, $\{A, D, E, G, T\}$. Its forward arcs are BA, BD, FG, FT.

$$\text{cap} = 45 + 65 + 25 + 55 = 190.$$

Part (c) gives a flow of 190, while this cut proves no flow can exceed 190. By the maximum-flow minimum-cut theorem, the two equal values prove optimality.

The maximum flow is 190 litres per minute.

(2)

(e) The new arc can add at most x to the former flow. The cut into T has unchanged capacity $85 + 70 + 55 = 210$, so it eventually becomes the restriction.

$$\text{maximum flow} = \begin{cases} 190 + x, & 0 \le x \le 20, \\ 210, & x \ge 20. \end{cases}$$

The maximum possible flow is $190 + x$ up to $x = 20$, then 210 litres per minute.

(3)

Question 4

Question 4

The unit costs, in hundreds of pounds, of transportation from suppliers A, B, C, D to depots X, Y, Z, with stocks and demands, are

	X	Y	Z	stock
A	13	12	11	15
B	15	8	10	12
C	11	12	14	8
D	11	13	12	14
demand	24	12	13	

The LP is to minimise

$$13AX + 12AY + 11AZ + 15BX + 8BY + 10BZ + 11CX + 12CY + 14CZ + 11DX + 13DY + 12DZ$$

subject to each supplier's total being its stock and each depot's total being its demand.

$$AX + AY + AZ = 15, \quad BX + BY + BZ = 12, \quad CX + CY + CZ = 8, \quad DX + DY + DZ = 14,$$
$$AX + BX + CX + DX = 24, \quad AY + BY + CY + DY = 12, \quad AZ + BZ + CZ + DZ = 13.$$

(a) Explain the purpose of (i) the objective line and (ii) the constraint $AX + BX + CX + DX = 24$. **(3)**

The solver output is $AX = 2, AZ = 13, BY = 12, CX = 8, DX = 14$, with every other variable zero. **(b)** (i) Interpret this output. (ii) Find the minimum total cost. **(2)**

Stock at B is later found to be 16, while demands and other stocks are unchanged. **(c)** Explain why replacing $BX + BY + BZ = 12$ by $BX + BY + BZ = 16$ alone is incorrect. **(1)**

(d) State the effect on (i) the objective function and (ii) the constraints. **(2)**

The modified solver run has minimum cost \$50300, and only AZ and BZ differ from the original solution. **(e)** Determine the new amounts transported from A and B to Z. **(3)**

Model solution

(a) Each variable is the number of units sent along the route denoted by its two letters. (i) Each coefficient is the route cost in hundreds of pounds, so the objective is the total transportation cost, which is minimised. (ii) $AX + BX + CX + DX = 24$ *makes the total delivered to depot* X *equal its demand of* 24 *units.*

The line is the minimum-cost objective; the stated constraint meets X*'s demand.*

(3)

(b) Read each non-zero solver variable as a delivery.

$A \to X : 2, A \to Z : 13, B \to Y : 12, C \to X : 8, D \to X : 14$, *with no other deliveries.*

$$2(13) + 13(11) + 12(8) + 8(11) + 14(11) = 507$$

This is in hundreds of pounds.

The minimum total cost is $507 \times \$100 = \$50700$.

(2)

(c) Total supply would become $15+16+8+14 = 53$*, but total demand remains* $24+12+13 = 49$*. Equalities would require all* 53 *units to be sent even though only* 49 *are wanted.*

The problem becomes unbalanced; there are four surplus units.

(1)

(d) Route costs have not changed, so the cost expression remains the same. Stock is now an upper bound because not all stock has to be transported.

$$AX+AY+AZ \leq 15, \quad BX+BY+BZ \leq 16, \quad CX+CY+CZ \leq 8, \quad DX+DY+DZ \leq 14.$$

(i) No change to the objective function. (ii) Replace the stock equalities by the displayed $\leq$ *constraints.*

(2)

(e) All variables other than AZ, BZ *are unchanged. The new objective, in hundreds of pounds, is* 503*. The unchanged deliveries cost* $507 - 13(11) = 364$*, leaving* 139 *for deliveries to* Z*. Demand at* Z *is still* 13*.*

$$11AZ + 10BZ = 139, \qquad AZ + BZ = 13.$$

$$11AZ + 10(13 - AZ) = 139 \implies AZ = 9, \qquad BZ = 4.$$

$AZ = 9$ *and* $BZ = 4$.

(3)

Question 5

Question 5

The initial tableau for a two-stage simplex solution of a maximisation LP is

Q	P	x	y	z	s_1	s_2	s_3	s_4	a_1	a_2	RHS
1	0	3	1	2	0	−1	−1	0	0	0	50
0	1	−2	−3	$-k$	0	0	0	0	0	0	0
0	0	1	3	1	1	0	0	0	0	0	30
0	0	0	2	1	0	−1	0	0	1	0	20
0	0	3	−1	1	0	0	−1	0	0	1	30
0	0	1	1	2	0	0	0	1	0	0	40

(**a**) Formulate the original LP. **(3)**
(**b**) Show how the first numerical row was formed. **(2)**
After three first-stage iterations, the tableau is

Q	P	x	y	z	s_1	s_2	s_3	s_4	a_1	a_2	RHS
1	0	0	0	0	0	0	0	0	−1	−1	0
0	1	0	0	0	$\frac{5}{2}-k$	$\frac{7}{3}-\frac{5}{3}k$	$\frac{1}{6}-\frac{1}{3}k$	0	$-\frac{7}{3}+\frac{5}{3}k$	$-\frac{1}{6}+\frac{1}{3}k$	$\frac{70}{3}+\frac{40}{3}k$
0	0	1	0	0	$\frac{1}{2}$	$\frac{2}{3}$	$-\frac{1}{6}$	0	$-\frac{2}{3}$	$\frac{1}{6}$	$\frac{20}{3}$
0	0	0	1	0	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	0	$-\frac{1}{3}$	$-\frac{1}{6}$	$\frac{10}{3}$
0	0	0	0	1	−1	$-\frac{5}{3}$	$-\frac{1}{3}$	0	$\frac{5}{3}$	$\frac{1}{3}$	$\frac{40}{3}$
0	0	0	0	0	1	$\frac{7}{3}$	$\frac{2}{3}$	1	$-\frac{7}{3}$	$-\frac{2}{3}$	$\frac{10}{3}$

(**c**) Explain why the first stage is complete. **(2)**
(**d**) Start the second stage and carry out one iteration using an entry in the s_2 column as pivot. **(4)**
Given that this is optimal after that iteration, (**e**) (i) determine the range of k; (ii) state the optimal x, y, z and maximum P. **(4)**

Model solution

(a) The P-row gives the objective and each constraint row is read after moving the slack, surplus and artificial variables out.

$$\begin{aligned} \textit{maximise } & P = 2x + 3y + kz \\ \textit{subject to } & x + 3y + z \leq 30, \\ & 2y + z \geq 20, \\ & 3x - y + z \geq 30, \\ & x + y + 2z \leq 40, \\ & x, y, z \geq 0. \end{aligned}$$

The LP is as displayed.

(3)

(b) In stage 1, $Q = a_1 + a_2$. Use the two rows containing the artificial variables to eliminate them from this objective.

$$Q = (20 - 2y - z + s_2) + (30 - 3x + y - z + s_3),$$

$$Q + 3x + y + 2z - s_2 - s_3 = 50.$$

This is the first numerical row of the initial tableau.

(2)

(c) *First-stage feasibility requires the artificial-variable objective to be zero. Here $Q = 0$. For a minimisation of Q, the remaining negative entries in its row are only in artificial-variable columns, which are discarded for stage 2.*

The first stage is complete: the solution is feasible and no further non-artificial improvement is possible.

(2)

(d) *Delete the Q, a_1, a_2 columns. In the P-row, s_2 has coefficient $\frac{7}{3} - \frac{5}{3}k$, so use the positive s_2-entries and the ratio test: $\frac{20/3}{2/3} = 10$, $\frac{10/3}{1/3} = 10$, $\frac{10/3}{7/3} = \frac{10}{7}$. Thus the last row leaves and s_2 enters.*

P	x	y	z	s_1	s_2	s_3	s_4	RHS
1	0	0	0	$\frac{3}{2} - \frac{2}{7}k$	0	$-\frac{1}{2} + \frac{1}{7}k$	$-1 + \frac{5}{7}k$	$20 + \frac{110}{7}k$
0	1	0	0	$\frac{3}{14}$	0	$-\frac{5}{14}$	$-\frac{2}{7}$	$\frac{40}{7}$
0	0	1	0	$\frac{5}{14}$	0	$-\frac{1}{14}$	$-\frac{1}{7}$	$\frac{20}{7}$
0	0	0	1	$-\frac{2}{7}$	0	$\frac{1}{7}$	$\frac{5}{7}$	$\frac{110}{7}$
0	0	0	0	$\frac{3}{7}$	1	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{10}{7}$

The displayed tableau is the result after the required second-stage iteration.

(4)

(e) *For the displayed basic feasible solution to be optimal in this maximisation tableau, the coefficients in the non-basic columns of the objective row must be non-negative.*

$$\frac{3}{2} - \frac{2}{7}k \geq 0, \qquad -\frac{1}{2} + \frac{1}{7}k \geq 0, \qquad -1 + \frac{5}{7}k \geq 0.$$

$$\frac{7}{2} \leq k \leq \frac{21}{4}.$$

Set the non-basic variables s_1, s_3, s_4 to zero and read the basic-variable values from the RHS.

$$x = \frac{40}{7}, \qquad y = \frac{20}{7}, \qquad z = \frac{110}{7}, \qquad P_{\max} = 20 + \frac{110}{7}k.$$

$\frac{7}{2} \leq k \leq \frac{21}{4}$, with $x = \frac{40}{7}, y = \frac{20}{7}, z = \frac{110}{7}$ and $P_{\max} = 20 + \frac{110}{7}k$.

(4)