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Physics and maths resources

OCR MEI A-Level Further Mathematics

H645/Y433 - Modelling with Algorithms

Tuesday 20 October 2020

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the supplied official OCR mark scheme. The original paper carries 60 marks. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

The numbers

$$12 \quad 15 \quad 11 \quad 9 \quad 20 \quad 16 \quad 8 \quad 23 \quad 10 \quad 7$$

represent masses, measured in kg.
(a) Apply the quick sort algorithm to sort the list into descending order. Use the first value as the pivot for each sublist.
(b) Show the result of applying the first fit decreasing algorithm to pack these masses into bags that can each hold a maximum of 40 kg.
(c) Show the result of applying a full bin strategy to pack these masses into bags that can each hold a maximum of 40 kg.

Model solution

(a) Quick sort places the first item of a current sublist in its final position, with larger values to its left because the required order is descending. Repeating this on each non-singleton sublist gives:

$$\begin{array}{cccccccccc} 12 & 15 & 11 & 9 & 20 & 16 & 8 & 23 & 10 & 7 \\ 15 & 20 & 16 & 23 & 12 & 11 & 9 & 8 & 10 & 7 \\ 20 & 16 & 23 & 15 & 12 & 11 & 9 & 8 & 10 & 7 \\ 23 & 20 & 16 & 15 & 12 & 11 & 10 & 9 & 8 & 7 \end{array}$$

All remaining sublists now contain one item, so the sort is complete.

$23, 20, 16, 15, 12, 11, 10, 9, 8, 7$, *in descending order.*

(3)

(b) First fit decreasing starts with the descending list above, and puts each mass in the first existing bag with enough remaining space.

bag	*contents (kg)*	*total (kg)*
1	$23, 16$	39
2	$20, 15$	35
3	$12, 11, 10, 7$	40
4	$9, 8$	17

The four bags are $(23, 16), (20, 15), (12, 11, 10, 7), (9, 8)$.

(2)

(c) A full-bin strategy looks for combinations that fill a bag exactly before moving on.

bag	*contents (kg)*	*total (kg)*
1	$23, 10, 7$	40
2	$20, 12, 8$	40
3	$16, 15, 9$	40
4	11	11

Three bags are full, and the remaining bag contains 11 *kg.*

(2)

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Question 2

Question 2

The table in Fig. 2 gives distances, in miles, along direct roads between the ten villages A to J. A dash indicates that there is no direct road.

	A	B	C	D	E	F	G	H	I	J
A	–	9	–	10	13	2	17	6	–	–
B	9	–	14	–	–	–	10	7	2	10
C	–	14	–	8	18	7	–	6	–	5
D	10	–	8	–	–	16	–	11	–	15
E	13	–	18	–	–	–	7	–	–	8
F	2	–	7	16	–	–	13	10	9	–
G	17	10	–	–	7	13	–	13	11	–
H	6	7	6	11	–	10	13	–	–	9
I	–	2	–	–	–	9	11	–	–	–
J	–	10	5	15	8	–	–	9	–	–

(a) Apply the tabular form of Prim's algorithm, starting at vertex D, to find a minimum spanning tree.
(b) State the total length of the arcs in the minimum spanning tree.
(c) Use Dijkstra's algorithm to determine the length of the shortest path from G to J via D.
(d) A computer takes 0.25 seconds to solve a shortest-path problem on a network with 15 vertices using Dijkstra's algorithm. Approximately how long will it take on a network with 300 vertices?

Model solution

(a) Prim's algorithm repeatedly chooses the shortest arc joining the current tree to a new vertex. Starting at D, the choices are recorded below; the number is the arc length.

vertex added	D	C	J	H	A	F	I	B	E	G
joining arc	–	$DC:8$	$CJ:5$	$CH:6$	$HA:6$	$AF:2$	$FI:9$	$IB:2$	$JE:8$	$EG:7$

One minimum spanning tree is $DC, CJ, CH, HA, AF, FI, IB, JE, EG$.

(2)

(b) Add the nine chosen arc lengths.

$$8+5+6+6+2+9+2+8+7=51.$$

The total length is 51 miles.

(1)

(c) Because the route must pass through D, find the shortest route from D to G and from D to J, then add the two independent lengths. Dijkstra's permanent labels from D are:

vertex	D	C	J	H	A	F	I	B	E	G
permanent distance from D	0	8	13	11	17	12	21	19	21	24

For example, $D \to C \to J$ has length $8 + 5 = 13$, and the shortest route from D to G has label 24.

length from G to J via $D = 24 + 13 = 37$.

37 miles.

(5)

(d) The standard implementation of Dijkstra's algorithm has quadratic time complexity, so the time scales with the square of the number of vertices.

$$t = 0.25\left(\frac{300}{15}\right)^2 = 0.25(20)^2 = 100.$$

Approximately 100 seconds.

(2)

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Question 3

Question 3

The diagram represents a system of pipes through which fluid can flow from two sources to two sinks. It also shows a cut α. Arc weights are pipe capacities in gallons per hour.

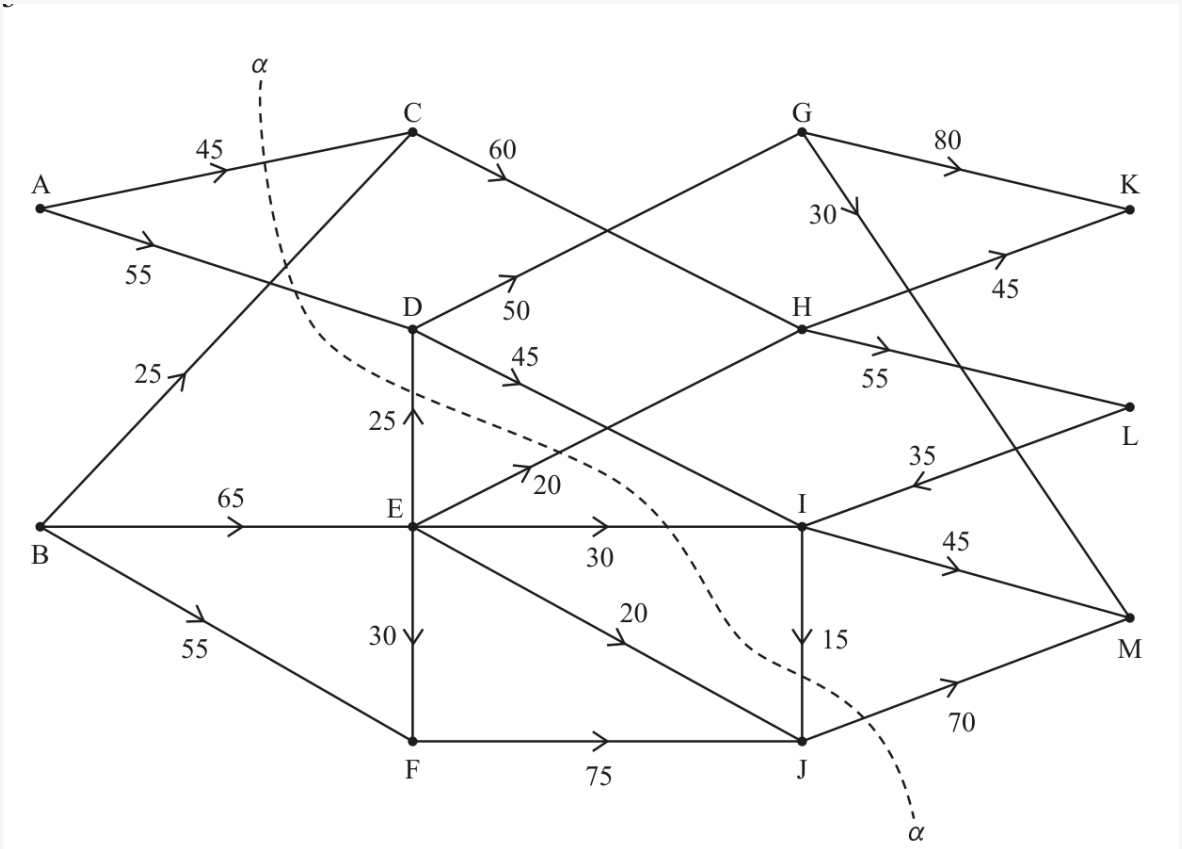


Fig. 3

The diagram in Fig. 3 represents a system of pipes through which a fluid can flow from two sources to two sinks. It also shows a cut α. The weights on the arcs show the capacities of the pipes in gallons per hour.

(a) Add a supersource and a supersink to the network in the Printed Answer Booklet. Give appropriate weightings and directions to the connecting arcs. **[2]**

(a) Add a supersource and a supersink. Give appropriate weights and directions to the connecting arcs.
(b) Calculate the capacity of cut α.
(c) Explain why arcs DG and DI cannot simultaneously be full to capacity.
An LP formulation is set up to find the maximum flow.
(d)(i) Write a suitable objective function.
(d)(ii) Write the constraint that ensures the flow into vertex I equals the flow out of I.
The complete LP was run in an LP solver. Arcs $AC, BE, BF, ED, CH, DG, HK, LI$ and JM, and possibly others, were saturated; the flow through DI was 5 gallons per hour.
(e) Determine the maximum flow.
(f) Use a suitable cut to prove that this is the maximum flow.

Model solution

(a) A supersource joins to each original source, and each original sink joins to a supersink. The connecting capacities must be at least the total possible flow at that source or sink.

$$S \to A : 100, \qquad S \to B : 145, \qquad K \to T : 125, \qquad M \to T : 145.$$

Add supersource S and supersink T with the directed arcs and capacities shown.

(2)

(b) The cut capacity is the sum of capacities of all arcs directed from the source side of the dashed cut to its sink side.

$$45 + 25 + 55 + 25 + 20 + 30 + 70 = 270.$$

270 *gallons per hour.*

(1)

(c) Compare the capacity entering D with the capacity required if both outgoing arcs were full.

$$AD + ED = 55 + 25 = 80, \qquad DG + DI = 50 + 45 = 95.$$

At most 80 can enter D, so flows of 50 and 45 on DG and DI are impossible simultaneously.

(1)

(d)(i) A maximum-flow objective can count the total flow leaving the two original source vertices.

Maximise $AC + AD + BC + BE + BF.$

Maximise $AC + AD + BC + BE + BF.$

(1)

(d)(ii) At an internal vertex, conservation of flow says inflow minus outflow is zero. The arcs entering I are DI, EI, LI, and the arcs leaving are IJ, IM.

$$DI + EI + LI - IJ - IM = 0.$$

$$DI + EI + LI - IJ - IM = 0.$$

(1)

(e) Use the given saturated arcs and conservation at each vertex. At C, $BC = 60 - 45 = 15$. At D, $AD = 50 + 5 - 25 = 30$. Since $LI = 35$, conservation at L gives $HL = 35$; hence

$EH = 45 + 35 - 60 = 20$. *A feasible completion is* $EI = 20$, $EF = EJ = 0$, $FJ = 55$, $IJ = 15$, *and* $IM = 5 + 20 + 35 - 15 = 45$. *Choose* $GK = 20$, $GM = 30$. *Every arc respects its capacity and each intermediate vertex conserves flow. The total source flow is:*

$$AC + AD + BC + BE + BF = 45 + 30 + 15 + 65 + 55 = 210.$$

The maximum flow is 210 *gallons per hour.*

(3)

(f) Take the cut with source-side vertices $\{A, B, C, D, E, F, H, I, J, L\}$ *and sink-side vertices* $\{G, K, M\}$. *Its forward-crossing arcs have capacities:*

$$DG + HK + IM + JM = 50 + 45 + 45 + 70 = 210.$$

Every flow is at most the capacity of any cut. Since part (e) constructs a flow of 210, *the max-flow min-cut theorem shows that this upper bound is attained.*

The maximum flow is 210 *gallons per hour.*

(2)

Question 4

Question 4

The LP formulation below finds the minimum completion time T, in hours, for the 11 activities $A, B, \ldots, K$.

Minimise T

$$T - I \geq 6,\ T - J \geq 11,\ T - K \geq 10,\ I - D \geq 10,\ J - D \geq 10,$$
$$I - H \geq 4,\ J - H \geq 4,\ K - D \geq 10,\ K - H \geq 4,\ K - F \geq 9,$$
$$K - G \geq 12,\ H - E \geq 11,\ F - E \geq 11,\ H - A \geq 5,\ F - A \geq 5,$$
$$G - A \geq 5,\ H - C \geq 7,\ F - C \geq 7,\ D - B \geq 8,\ E - B \geq 8.$$

(a) Give the duration and immediate predecessors of each activity.
(b) Draw an activity-on-arc network for the project.
(c) Carry out a forward pass and backward pass, showing early and late event times at each vertex.
Activity C has a non-zero independent float.
(d)(i) State the other activity with non-zero independent float.
(d)(ii) Calculate the corresponding independent float.
(d)(iii) Explain the significance for delays in the project.

Model solution

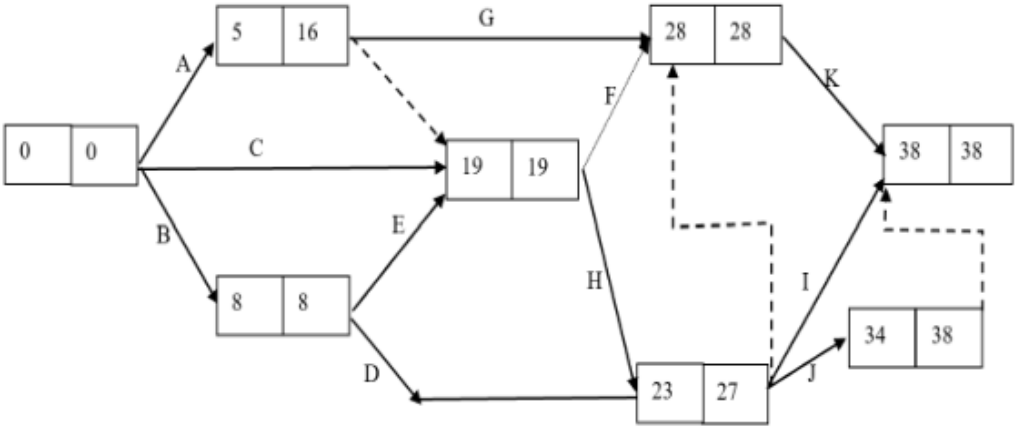
(a) In a constraint such as $F - E \geq 11$, activity E is an immediate predecessor of F, and 11 is the duration of E. Reading each constraint this way gives:

activity	*duration*	*immediate predecessors*
A	5	–
B	8	–
C	7	–
D	10	B
E	11	B
F	9	A, C, E
G	12	A
H	4	A, C, E
I	6	D, H
J	11	D, H
K	10	D, F, G, H

The durations and immediate predecessors are as tabulated.

(3)

(b) An activity-on-arc network needs dummy arcs wherever an event must wait for selected predecessors without representing an extra activity. The following network has one start, one finish, and three directed dummy arcs (dashed). Each real arc is labelled by its activity.



The displayed activity-on-arc network represents all the predecessor requirements.

(3)

(c) *For a forward pass, each event time is the largest of its incoming completion times. For a backward pass, each late event time is the smallest of its outgoing latest-start times.*

event	1	2	3	4	5	6	7	8
early time	0	5	8	19	23	28	34	38
late time	0	16	8	19	27	28	38	38

For instance, the event after D and H has early time $\max(8+10, 19+4) = 23$. The backward pass then gives its late time as 27.

The project completion time is 38 hours.

(4)

(d)(i) *Independent float is checked from the early time at the end event minus the late time at the start event and the activity duration.*

Activity I.

(1)

(d)(ii) *For I, use the early time at its end event, the late time at its start event, and its duration.*

$$\textit{independent float} = 38 - 27 - 6 = 5.$$

5 hours.

(1)

(d)(iii) *Independent float is delay available without delaying the early start of any successor and without using the float of a predecessor.*

Activity I may be delayed by 5 hours without affecting any other activity or the 38-hour completion time. ***(1)***

Question 5

Question 5

Fig. 5.1 shows the constraints of a linear-programming problem whose objective is to maximise a function of x and y. The feasible region R is the unshaded region, including its boundaries. An objective line is shown.

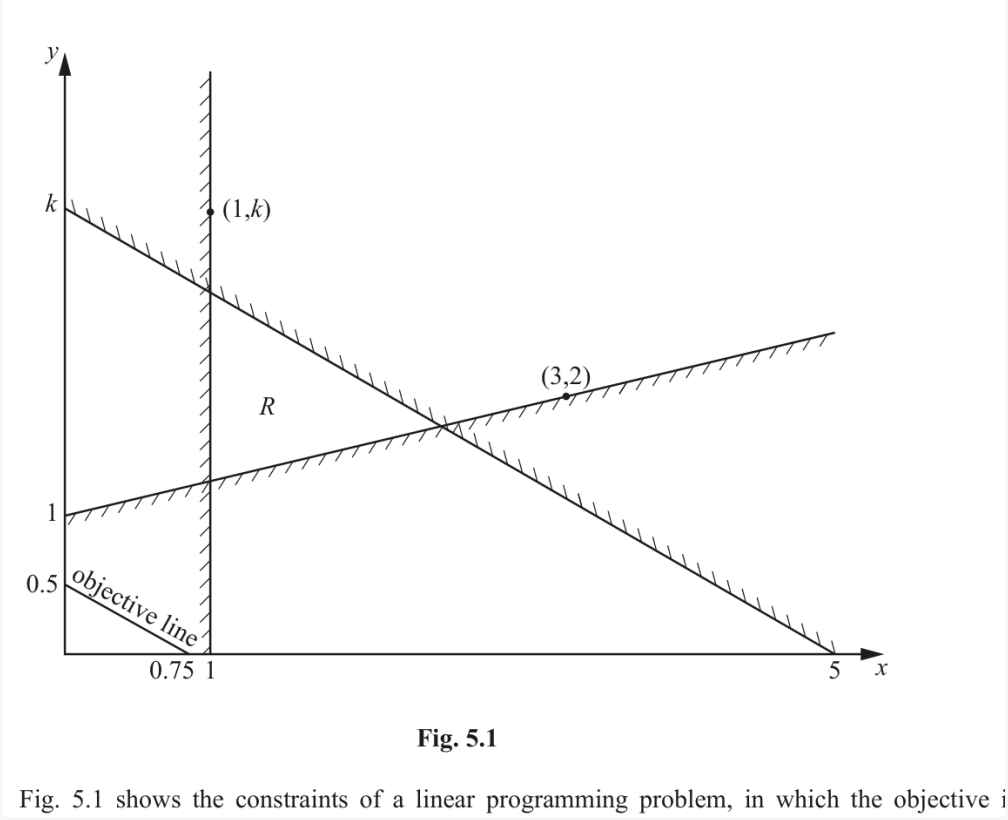


Fig. 5.1

Fig. 5.1 shows the constraints of a linear programming problem, in which the objective i

(a) Complete an initial tableau so that the two-stage simplex method can be used. Show the equations using slack, surplus and artificial variables, and show how the two objective rows are formed. After two iterations, a computer produces the tableau

Q	P	x	y	s_1	s_2	s_3	a_1	a_2	RHS
1	0	0	0	0	0	0	-1	-1	0
0	1	0	0	-3	-1	0	3	1	6
0	0	1	0	-1	0	0	1	0	1
0	0	0	1	$-\frac{1}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{4}{3}$
0	0	0	0	$k+\frac{5}{3}$	$\frac{5}{3}$	1	$-\left(k+\frac{5}{3}\right)$	$-\frac{5}{3}$	$4k-\frac{20}{3}$

It does not give an optimal solution.

(b) Reduce the tableau for the second stage, then carry out one iteration using an entry in the s_2 column as pivot.

An optimum is not found after this iteration.

(c) Determine the range of k.

(d) Given that the y-coordinate of the optimal vertex is $\frac{12}{7}$, determine k.

Model solution

(a) The objective line has slope $-\frac{2}{3}$, so a suitable objective is $P = 2x + 3y$. From the three boundary lines and their shaded sides, the constraints are $x \geq 1$, $3y \geq x + 3$, and $5y + kx \leq 5k$. The first two need a surplus and artificial variable; the last needs a slack variable.

$$x - s_1 + a_1 = 1,$$
$$-x + 3y - s_2 + a_2 = 3,$$
$$kx + 5y + s_3 = 5k.$$

For stage 1, minimise the artificial-variable total $Q = a_1 + a_2$. Substituting from the first two equations gives $Q + 3y - s_1 - s_2 = 4$. The stage-2 objective row is $P - 2x - 3y = 0$.

Q	P	x	y	s_1	s_2	s_3	a_1	a_2	RHS
1	0	0	3	−1	−1	0	0	0	4
0	1	−2	−3	0	0	0	0	0	0
0	0	1	0	−1	0	0	1	0	1
0	0	−1	3	0	−1	0	0	1	3
0	0	k	5	0	0	1	0	0	$5k$

The equations and initial two-stage tableau are as shown.

(10)

(b) First remove the artificial-variable columns and the phase-1 objective row. The columns P, x, y, s_3 form the current basis. To pivot in the s_2 column, the positive entry $\frac{5}{3}$ in the final row is used; this makes s_3 leave the basis.

P	x	y	s_1	s_2	s_3	RHS
1	0	0	−3	−1	0	6
0	1	0	−1	0	0	1
0	0	1	$-\frac{1}{3}$	$-\frac{1}{3}$	0	$\frac{4}{3}$
0	0	0	$k + \frac{5}{3}$	$\frac{5}{3}$	1	$4k - \frac{20}{3}$

Divide the final row by $\frac{5}{3}$, then eliminate s_2 from the other rows.

P	x	y	s_1	s_2	s_3	RHS
1	0	0	$\frac{3}{5}k - 2$	0	$\frac{3}{5}$	$2 + \frac{12}{5}k$
0	1	0	−1	0	0	1
0	0	1	$-\frac{1}{5}k$	0	$\frac{1}{5}$	$\frac{4}{5}k$
0	0	0	$\frac{3}{5}k + 1$	1	$\frac{3}{5}$	$\frac{12}{5}k - 4$

The reduced and once-pivoted tableaus are as shown.

(4)

(c) After the pivot, the objective row still has to show an improving direction, because the question says the solution is not yet optimal. Its s_1 coefficient must therefore be negative.

$$\frac{3}{5}k - 2 < 0 \quad \Longrightarrow \quad k < \frac{10}{3}.$$

The position of the feasible region in the graph supplies the other restriction, $k > 2$.

$$2 < k < \tfrac{10}{3}.$$

(3)

(d) At the optimum, the two sloping boundaries meet. The lower boundary is $3y = x + 3$, so the given y-coordinate first gives x.

$$x = 3\left(\frac{12}{7}\right) - 3 = \frac{15}{7}.$$

Substitute this point into $kx + 5y = 5k$.

$$k\left(\frac{15}{7}\right) + 5\left(\frac{12}{7}\right) = 5k \implies 15k + 60 = 35k \implies k = 3.$$

$$k = 3.$$

(2)

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