Maths with Matt

Physics and maths resources

OCR MEI A-Level Further Mathematics

H645/Y434 - Numerical Methods

Thursday 20 June 2019

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the supplied official OCR mark scheme. The original paper carries 60 marks. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

Figure 1 is spreadsheet output for $f(x) = e^{-x^2}$. Cells B2–B6 contain $f(1), \ldots, f(5)$, and C2 is $=$ B2, while C3 is $=$ B2 $+$ C2, with equivalent formulae down the columns.

x	1	2	3	4	5
$f(x)$	0.367879441	0.018315639	0.000123410	1.12535E−7	1.38879E−11
running total	0.367879441	0.386195080	0.386318490	0.386318602	0.386318602

(a) Express the formula in C5 using sigma notation. **(2)**
(b) Explain why the same value is displayed in C5 and C6. **(2)**
Now chop the value in C2 to 3 decimal places and use it to approximate C2. (c) Calculate the relative error. **(1)**
(d) The values in B4, B5 and B6 are each chopped to 3 decimal places. Explain why the relative errors are all the same. **(1)**

Model solution

(a) C5 is the total of the entries in B2 through B5, which correspond to $i = 1$ through 4.

$$\text{C5} = \sum_{i=1}^{4} e^{-i^2}.$$

$$\sum_{i=1}^{4} e^{-i^2}$$

(2) (b) The next term added is $e^{-25} = 1.38879 \times 10^{-11}$. It is too small to change the displayed running total, which is shown to 9 decimal places.

C5 and C6 round to the same displayed value.

(2) (c) Chopping gives 0.367. Relative error is taken as (approximation − true value/true value).

$$\frac{0.367 - 0.367879441}{0.367879441} = -0.002390568\ldots$$

-0.00239 *(approximately).*

(1) (d) Each of B4, B5 and B6 is smaller than 0.001, so chopping to 3 decimal places makes each approximation 0. Thus each relative error is $(0 - v/v = -1)$.

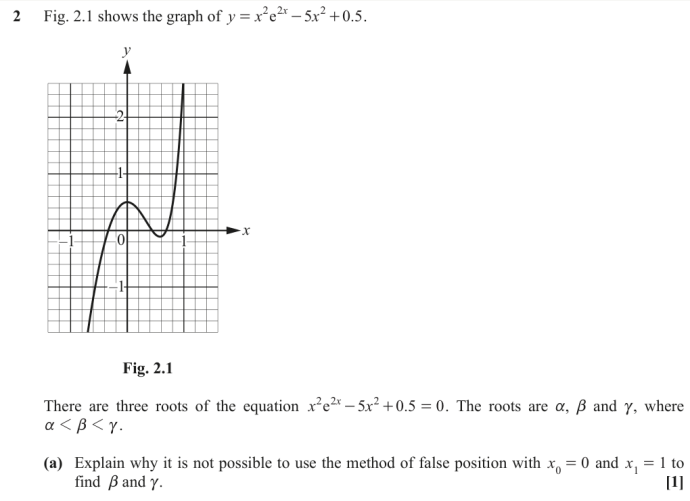
All are -1 (or have magnitude 1).

(1)

Question 2

Question 2

Figure 2.1 shows $y = x^2e^{2x} - 5x^2 + 0.5$. Its three roots are $\alpha < \beta < \gamma$.



(a) Explain why false position with $x_0 = 0$ and $x_1 = 1$ cannot find β and γ. **(1)**

For γ, the starting interval is $[0.6, 0.8]$. The false-position spreadsheet has final rows

a	$f(a)$	b	$f(b)$	approx	f(approx)
0.684692	−0.00025	0.8	0.469941	0.684753	−0.00011
0.684753	−0.00011	0.8	0.469941	0.684780	−5.1E−5

(b) Without further calculation, give the smallest interval certain to contain γ. **(1)**

(c) State what column F calculates. **(1)**

Cell A3 has formula $=$ IF(F2 <0, E2, A2). (d) Explain its purpose. **(2)**

(e) Explain how false position differs from the secant method. **(1)**

(f) Give one advantage and one disadvantage of false position rather than secant. **(2)**

Model solution

(a) False position needs an initial sign change so that the root is bracketed. Here $f(0) = 0.5$ and $f(1) = e^2 - 4.5$, both positive.

There is no sign change between 0 and 1.

(1) (b) The last approximation is negative and $f(0.8) > 0$, so these endpoints still bracket the root.

$$0.684780 < \gamma < 0.8.$$

(1) (c) Once the chord intercept has been calculated, its function value determines which endpoint is replaced.

Column F calculates f(new approximation).

(1) (d) It tests the sign of f(approx), preserving a sign-changing interval. If F2 is

negative it copies E2 to A3; otherwise it copies A2.

It selects the correct left endpoint for the next bracketing interval.

(2) *(e) Both use the x-intercept of a secant chord, but false position retains endpoints of opposite sign; the secant method does not require this.*

False position brackets the root; secant iteration need not.

(1) *(f) Maintaining a bracket means false position is reliably convergent to a root. That same restriction can make it slower than secant.*

Advantage: it keeps the root bracketed and converges. Disadvantage: it can converge more slowly.

(2)

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Question 3

Question 3

In week (n), the cumulative total (y) diagnosed with influenza was

n	1	2	3	4	5
y	9	32	63	96	125

.

(a) Complete the difference table. **(3)**
(b) Explain why a cubic model is appropriate. **(1)**
(c) Use Newton's method to find the interpolating polynomial of degree 3. **(4)**
In weeks 6 and 7, the actual cumulative total was 145. (d) Determine whether the model is a good fit. **(2)**
(e) Determine the maximum number of weeks for which the model could possibly be valid. **(1)**

Model solution

(a) Successive differences reveal the degree of a polynomial model.

n	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	9	23	8	-6
2	32	31	2	-6
3	63	33	-4	
4	96	29		
5	125			

Difference table as shown.

(3) (b) The third differences are constant (equivalently, fourth differences would be zero), the characteristic pattern for a cubic.

A cubic is appropriate.

(1) (c) Newton's forward formula uses the first entry in each difference column, with origin $n = 1$:

$$y = 9 + 23(n-1) + \frac{8(n-1)(n-2)}{2!} + \frac{-6(n-1)(n-2)(n-3)}{3!}.$$

$$y = 10n^2 - n^3.$$

$$y = 10n^2 - n^3.$$

(4) (d) Evaluate the model at the two extra weeks, then compare directly with the recorded total.

$$y(6) = 144, \qquad y(7) = 147.$$

Both predictions are close to 145, so it is a good fit for these data.

(2) *(e) The model must not predict a negative number of patients. Since* $y = n^2(10 - n)$*, it becomes negative after* $n = 10$*.*

At most 10 weeks.

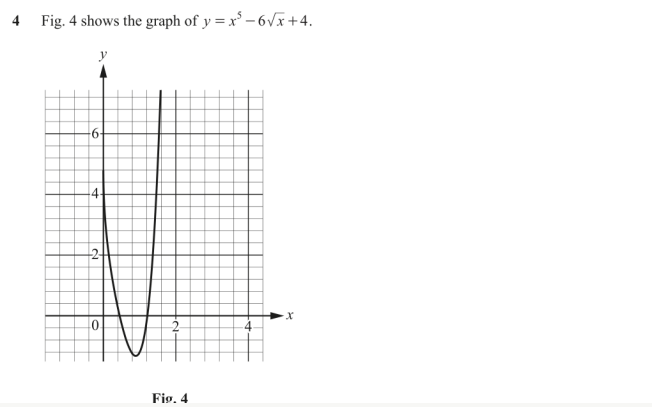
(1)

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Question 4

Question 4

Figure 4 shows $y = x^5 - 6\sqrt{x} + 4$. Its roots are $a < b$.



(a) Show that $0 < a < 1$ and $1 < b < 2$. **(2)**

(b) Obtain the Newton–Raphson iteration $x_{n+1} = x_n - \dfrac{x_n^{11/2} - 6x_n + 4\sqrt{x_n}}{5x_n^{9/2} - 3}$. **(3)**

(c) Starting with $x_0 = 1$, obtain (b) to 6 decimal places. **(2)**

(d) Starting with $x_0 = 0$, find x_1. **(1)**

(e) Give a geometrical explanation for this failure to find (a). **(1)**

(f) Obtain $x_{n+1} = \left(\dfrac{x_n^5 + 4}{6}\right)^2$. **(2)**

(g) Starting with $x_0 = 0$, obtain (a) to 6 decimal places. **(2)**

Model solution

(a) A sign change on each stated interval establishes a root there.

$$f(0) = 4, \quad f(1) = -1, \quad f(2) = 32 - 6\sqrt{2} + 4 > 0.$$

$$0 < a < 1 \text{ and } 1 < b < 2.$$

(2) (b) Differentiate before applying Newton--Raphson, $x_{n+1} = x_n - f(x_n)/f'(x_n)$, then multiply the fraction's numerator and denominator by $\sqrt{x_n}$.

$$f'(x) = 5x^4 - 3x^{-1/2}, \qquad x_{n+1} = x_n - \frac{x_n^5 - 6\sqrt{x_n} + 4}{5x_n^4 - 3x_n^{-1/2}} = x_n - \frac{x_n^{11/2} - 6x_n + 4\sqrt{x_n}}{5x_n^{9/2} - 3}.$$

$$x_{n+1} = x_n - \frac{x_n^{11/2} - 6x_n + 4\sqrt{x_n}}{5x_n^{9/2} - 3}.$$

(3) (c) Iterate, retaining guard figures.

$$1, \ 1.5, \ 1.3143166, \ 1.2295168, \ 1.2115408, \ 1.2107830, \ 1.2107817.$$

$$b = 1.210782.$$

(2) (d) At $x_0 = 0$, the vertical tangent gives a limiting Newton--Raphson step of zero.

$$x_1 = 0.$$

(1) (e) At $x = 0$ the curve has a vertical tangent. Its tangent meets the x-axis again at the same x-coordinate, so Newton--Raphson repeats 0.

The vertical tangent causes the iteration to remain at 0.

(1) (f) Rearrange the root equation to isolate the square root, then square.

$$\sqrt{x} = \frac{x^5 + 4}{6} \quad \Rightarrow \quad x = \left(\frac{x^5 + 4}{6}\right)^2.$$

$$x_{n+1} = \left(\frac{x_n^5 + 4}{6}\right)^2.$$

(2) (g) Fixed-point iteration now gives

$$0,\ 0.4444444,\ 0.4483065,\ 0.4484776,\ 0.4484853,\ 0.4484857.$$

$$a = 0.448486.$$

(2)

Question 5

Question 5

The following estimates use the forward difference method for $f'(2)$.

h	estimate	difference	ratio
0.1	6.3050005		
0.01	6.0300512	−0.274949	
0.001	6.0030018	−0.027049	0.098379
0.0001	6.0003014	−0.002700	0.099835
0.00001	6.0000314	−0.000270	0.099983
0.000001	6.0000044	−2.7E−5	0.099994
0.0000001	6.0000016	−2.71E−6	0.100352
0.00000001	6.0000013	−3.02E−7	0.111457
0.000000001	6.0000018	4.885E−7	−1.617650
0.0000000001	6.0000049	3.109E−6	6.363636
0.00000000001	6.0000005	−4.44E−6	−1.428570
0.000000000001	6.0005334	0.0005329	−120
0.0000000000001	5.9952043	−0.005329	−10
0.00000000000001	6.1284311	0.1332268	−25
0.000000000000001	5.3290705	−0.799361	−6
0.0000000000000001	0	−5.329071	6.666667

(a) Give formulae for C3 and D4. **(2)**
(b) Explain what D4–D8 show about order of convergence. **(2)**
(c) Write A10 in standard notation. **(1)**
(d) Explain what D10–D17 suggest about B10–B16. **(2)**
(e) Give $f'(2)$ to justified accuracy. **(2)**
The B-column formula, with A2 $= h$, is equivalent to

$$\frac{\ln\sqrt{\sinh((2+\text{A2})^3)}}{\text{A2}} - \frac{\ln\sqrt{\sinh(2^3)}}{\text{A2}}.$$

(f) Write $f(x)$. **(1)**
(g) Explain why B17 displays zero although the calculated answer is non-zero. **(2)**

Model solution

(a) A difference subtracts successive estimates; a ratio compares successive differences.

C3: $= \text{B3} - \text{B2}$*; D4:* $= \text{C4}/\text{C3}$.

(2) (b) The ratios are close to 0.1*, matching the factor by which (h) is reduced. Hence the error is proportional to (h).*

The method is first-order convergent.

(2)

(c) 1×10^{-9}.

(1) (d) The ratios cease to settle and diverge. Round-off and cancellation are dominating, so the estimates near the bottom are becoming less accurate.

B10--B16 are not increasingly reliable.

(2) (e) The estimates agree through 6.000001 *before numerical errors become evident; this is the precision justified by the data.*

$f'(2) \approx 6.000001$ *(or* 6.00000 *certainly).*

(2) (f) The formula is $[f(2+h) - f(2)]/h$*, so read off the underlying function.*

$f(x) = \ln(\sqrt{\sinh(x^3)})$.

(1) (g) For tiny (h), the numerator subtracts two nearly equal stored values. At spreadsheet precision they become identical non-zero numbers, so their subtraction displays zero.

Catastrophic cancellation/finite precision has produced the displayed zero.

(2)

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Question 6

Question 6

The spreadsheet gives midpoint-rule estimates M_n and trapezium-rule estimates T_n for

$$\int_0^1 x^{-x^x}\,\mathrm{d}x.$$

n	M_n	T_n
1	1.632527	1
2	1.641461	1.316263
4	1.623053	1.478862
8	1.610295	1.550957
16	1.604132	1.580626
32	1.601505	1.592379

(a) Write an efficient spreadsheet formula for C3. **(2)**

(b) First complete the Printed Answer Booklet table using Simpson's rule. Calculate the most accurate estimate possible, justifying the quoted precision. **(8)**

Model solution

(a) When the number of trapezia is doubled, the new trapezium estimate is the mean of the previous trapezium estimate and the corresponding midpoint estimate.

C3: $= (\mathrm{B2} + \mathrm{C2})/2$.

(2) (b) Simpson's rule gives $S_{2n} = (2M_n + T_n)/3$. *Applying it gives the following estimates; differences and ratios diagnose the remaining error.*

S_{2n}	*difference*	*ratio*
1.421685		
1.533062	0.1113771	
1.574989	0.0419276	0.376447
1.590516	0.0155267	0.370321
1.596297	0.0057806	0.372299
1.598463	0.0021665	0.374787

The ratios tend to about $r = 0.375$. *Extrapolating the last estimate gives*

$$I \approx 1.598463 + \frac{0.374787(0.0021665)}{1 - 0.374787} = 1.59976\ldots$$

This supports 1.60 *as certain;* 1.600 *is only probable.*

Most accurate justified estimate: $\boxed{1.60}$.

(8)