Maths with Matt

Physics and maths resources

OCR MEI A-Level Further Mathematics

H645/Y434 - Numerical Methods

Wednesday 21 October 2020

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the supplied official OCR mark scheme. The original paper carries 60 marks. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

Fig. 1 shows some spreadsheet output.

A	
1	1E−17
2	1E−17
3	1E−29

A3 = A2 − A1.
(a) Write the value displayed in cell A3 in standard mathematical notation.
(b) Explain why the value displayed in cell A3 is non-zero.
(c) Write down the value of the number stored in cell A2 to the highest precision possible.
(d) Explain why your answer to part (c) may be different to the actual value stored in cell A2.

Model solution

(a) Spreadsheet notation $\mathrm{E}{-}29$ *means "multiply by* 10^{-29}*".*

$$1\mathrm{E}{-}29 = 1 \times 10^{-29}.$$

$$1 \times 10^{-29}$$

([1])

(b) Although A1 and A2 look identical on the display, the subtraction has exposed a small difference between the stored numbers.

The numbers stored in A1 and A2 are different; only their displayed values are the same.

([1])

(c) Add the displayed difference to A1. This gives every digit that can be determined from the displayed information.

$$A2 = 10^{-17} + 10^{-29} = 1.000\,000\,000\,001 \times 10^{-17}.$$

$$1.000\,000\,000\,001 \times 10^{-17}$$

([1])

(d) The display is rounded or truncated to a finite number of significant figures, so it does not reveal all the stored digits.

There may be further non-zero digits beyond the precision of the display.

([1])

Question 2

Question 2

Fig. 2 shows three values of x and the associated values of a function $f(x)$.

x	1	2	5
$f(x)$	5	16.6	76.6

Find a polynomial $p(x)$ of degree 2 to approximate $f(x)$, giving your answer in the form $p(x) = ax^2 + bx + c$, where a, b and c are constants to be determined.

Model solution

There are three data points, so a quadratic interpolant is appropriate. Lagrange interpolation builds a polynomial which has the required value at each of the three x-coordinates.

$$p(x) = 5\frac{(x-2)(x-5)}{(1-2)(1-5)} + 16.6\frac{(x-1)(x-5)}{(2-1)(2-5)} + 76.6\frac{(x-1)(x-2)}{(5-1)(5-2)}.$$

$$\begin{aligned} p(x) &= \frac{5}{4}(x^2 - 7x + 10) - \frac{83}{15}(x^2 - 6x + 5) + \frac{383}{60}(x^2 - 3x + 2) \\ &= 2.1x^2 + 5.3x - 2.4. \end{aligned}$$

$$p(x) = 2.1x^2 + 5.3x - 2.4$$

([5])

Question 3

Question 3

At Heathwick airport each passenger's luggage is weighed before being loaded into the hold of the aeroplane. Each weight is displayed digitally in kg to 1 decimal place. Some examples are 17.2, 19.9, 22.3, 20.1, 21.5.
On each flight, the total weight of luggage is calculated to ensure compliance with health and safety regulations. Winston assumes that displayed weights are rounded to 1 decimal place and that the total is calculated using the displayed values. On a flight to Athens there are 154 items of luggage.
(a) Determine the maximum possible error, according to Winston's model, when the total weight is calculated.
Piotre instead assumes the displayed weights are chopped to 1 decimal place, and that the total is calculated using the displayed values.
(b) Determine the maximum possible error, according to Piotre's model, when the total weight is calculated.
The total of the displayed weights is 3080.2 kg. When all the luggage is weighed together in the loading bay, the total is 3089.44 kg.
(c) Determine whether Winston's model or Piotre's model is a better fit for the data.

Model solution

(a) Rounding to the nearest 0.1 *kg makes the error in one displayed weight at most* 0.05 *kg. In the worst case all errors have the same sign.*

$$154 \times 0.05 = 7.7 \text{ kg}.$$

Maximum possible error $= \pm 7.7$ *kg.*

([2])

(b) Chopping always removes up to 0.1 *kg from each item, so the total can be underestimated by almost* 154 *such amounts.*

$$154 \times 0.1 = 15.4 \text{ kg}.$$

Maximum possible error $= 15.4$ *kg (an underestimate).*

([2])

(c) The observed shortfall is $3089.44 - 3080.2 = 9.24$ *kg. Compare this with each model's largest possible shortfall.*

$$3080.2 + 7.7 = 3087.9 < 3089.44, \qquad 3080.2 + 15.4 = 3095.6 > 3089.44.$$

Winston's model cannot produce the observed total; Piotre's model is the better fit.

([3])

Question 4

Question 4

(a) Use the trapezium rule with 1 strip to calculate an estimate of $\int_1^2 \sqrt{1+x^3}\,\mathrm{d}x$, giving your answer correct to six decimal places.
Fig. 4 gives further approximations to this integral, using the trapezium rule T_n and Simpson's rule S_{2n}.

n	T_n	S_{2n}
1		2.130135
2	2.149378	
4	2.134751	2.129862
8	2.131084	

(b) Write down an efficient formula for cell C4.
(c) Find S_4, giving your answer correct to 6 decimal places.
(d) Without doing any further calculation, state the value of the integral as accurately as possible, justifying the precision quoted.
(e) Use the fact that Simpson's rule is a fourth-order method to obtain an improved approximation, stating the value to a precision which seems justified.

Model solution

(a) With one strip, the interval width is $h = 2 - 1 = 1$. The trapezium rule uses the two endpoint ordinates.

$$T_1 = \frac{1}{2}\left(\sqrt{1+1^3} + \sqrt{1+2^3}\right) = \frac{1}{2}(\sqrt{2}+3) = 2.207107\ldots$$

$$2.207107$$

([2])

(b) Combining two trapezium estimates by Richardson extrapolation gives Simpson's estimate $S_{2n} = (4T_{2n} - T_n)/3$. Here C4 uses B5, which is T_8, and B4, which is T_4.

```
=(4*B5-B4)/3
```

([2])

(c) Apply the same formula using T_4 and T_2.

$$S_4 = \frac{4(2.134751) - 2.149378}{3} = 2.129875\ldots$$

$$S_4 = 2.129875$$

([2])

(d) The two Simpson estimates agree to four decimal places: $S_4 = 2.129875\ldots$ and $S_8 = 2.129862$. Thus four decimal places are secure; the fifth is plausible but has changed.

$$\int_1^2 \sqrt{1+x^3}\,\mathrm{d}x \approx 2.1299$$ ***(secure to 4 d.p.).***

([2])

(e) A fourth-order error is divided by $2^4 = 16$ ***when the step size is halved. Extrapolating*** S_4 ***and*** S_8 ***therefore removes the leading error term.***

$$I \approx S_8 + \frac{S_8 - S_4}{16-1} = 2.129862 + \frac{2.129862 - 2.129875}{15} = 2.129861\ldots$$

2.12986 ***(justified to 5 d.p.).***

([2])

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Question 5

Question 5

You are given that $g(x) = \dfrac{\sqrt[3]{x^x + 25}}{2}$. Fig. 5.1 gives:

x	1.45	1.55
$g(x)$	1.49468	1.49949

(a) Use the central difference method to calculate an estimate of $g'(1.5)$, giving your answer correct to 3 decimal places.
The equation $x^x - 8x^3 + 25 = 0$ has two roots, α and β, such that $\alpha \approx 1.5$ and $\beta \approx 4.4$.
(b) Obtain the iterative formula $x_{n+1} = g(x_n) = \dfrac{\sqrt[3]{x_n^{x_n} + 25}}{2}$.
(c) Use your answer to part (a) to explain why it is possible that this iterative formula may be used to find α.
(d) Starting with $x_0 = 1.5$, use the iterative formula to find x_1, x_2, x_3, x_4, x_5 and x_6.
(e) Use your answer to part (d) to state α correct to 8 decimal places.
Starting with $x_0 = 4.5$, the same iteration gives:

n	x_n
0	4.5
1	4.81826433
2	6.27473453
3	23.2937196
4	2.0654E+10
5	`#NUM!`

(f) Explain why `#NUM!` is displayed for x_5.
(g) On the diagram in the Printed Answer Booklet, starting with $x_0 = 4.5$, illustrate how the iterative formula works to find x_1 and x_2.
(h) Determine what happens when the relaxed iteration $x_{n+1} = (1 - \lambda)x_n + \lambda g(x_n)$ is used to try to find β, with $x_0 = 4.5$, in each case $\lambda = 0.5$ and $\lambda = -0.4$.

Model solution

(a) A central difference uses equally spaced values on both sides of 1.5, so it estimates the tangent gradient by the chord gradient.

$$g'(1.5) \approx \frac{g(1.55) - g(1.45)}{1.55 - 1.45} = \frac{1.49949 - 1.49468}{0.10} = 0.0481.$$

$$g'(1.5) \approx 0.048$$

([2])

(b) A fixed point satisfies $x = g(x)$. Rearranging the given equation isolates x in exactly that form.

$$x^x - 8x^3 + 25 = 0 \implies 8x^3 = x^x + 25 \implies x = \frac{\sqrt[3]{x^x + 25}}{2}.$$

$$x_{n+1} = \frac{\sqrt[3]{x_n^{x_n} + 25}}{2}$$

([2])

(c) A fixed-point iteration can converge locally when $|g'(\alpha)| < 1$. Since $\alpha \approx 1.5$ and the estimate is 0.048, this condition is plausible.

$-1 < g'(1.5) \approx 0.048 < 1$, *so convergence to α is possible.*

([1])

(d) Substitute each current value into g, retaining extra digits so the final rounding is reliable.

n	x_n
1	1.49697756786
2	1.49683286384
3	1.49682595579
4	1.49682562605
5	1.49682561031
6	1.49682560956

$$x_1 = 1.49697756786, \ldots, x_6 = 1.49682560956$$

([2])

(e) The iterates have stabilised, so round their common value to eight decimal places.

$$\alpha = 1.49682561$$

([1])

(f) At the next step the calculation needs a number of the form $x_4^{x_4}$. With $x_4 \approx 2.0654 \times 10^{10}$, this is too large for the spreadsheet to represent.

The required number is too large for the spreadsheet to display or store.

([1])

(g) A fixed-point (staircase) construction alternates vertically to $y = g(x)$ and horizontally to $y = x$. From $x_0 = 4.5$, mark $(4.5, 4.81826433)$, then move horizontally to $(4.81826433, 4.81826433)$; repeat to reach $(4.81826433, 6.27473453)$, then horizontally to $(6.27473453, 6.27473453)$.

The staircase gives $x_1 \approx 4.81826433$ and $x_2 \approx 6.27473453$.

([1])

(h) Substitute into the relaxed formula and follow the iterates.

λ	x_0	x_1	x_2	x_3	x_4	*behaviour*
-0.4	4.5	4.37269427	4.38170607	4.38181972	4.38182045	*converges to* $\beta \approx 4.382$
0.5	4.5	4.65913217	5.07496886	6.44804950	16.95516397	*diverges*

For $\lambda = -0.4$, the iteration converges to $\beta \approx 4.382$; for $\lambda = 0.5$, it diverges.

([3])

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Question 6

Question 6

Fig. 6.1 shows the graph of $y = e^{3x} - 11x - 0.5$ for $-0.5 \leq x \leq 1$.

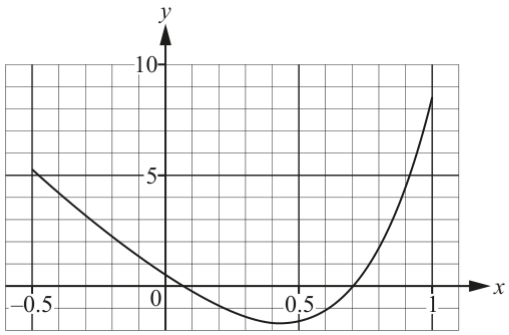


Fig. 6.1

he equation $e^{3x} - 11x - 0.5 = 0$ has two roots, α and β, such that $\alpha < \beta$. Dennis is going to
ıe method of interval bisection with starting values denoted by a and b.

a) Explain why the method of interval bisection starting with $a = 0$ and $b = 1$ may not be u
to find either α or β.

ennis uses the method of interval bisection starting with $a = 0$ and $b = 0.5$ to find α. So
preadsheet output is shown in Fig. 6.2.

The equation $e^{3x} - 11x - 0.5 = 0$ has roots α and β, where $\alpha < \beta$. Dennis uses interval bisection.
(a) Explain why starting with $a = 0$ and $b = 1$ may not be used to find either root.
Dennis starts with $a = 0$, $b = 0.5$ to find α. His spreadsheet columns are $a, f(a), b, f(b), x_{\text{new}}, f(x_{\text{new}})$, and rows include:

	a	$f(a)$	b	$f(b)$	x_{new}	$f(x_{\text{new}})$
2	0	0.5	0.5	−1.51831	0.25	−1.133
3	0	0.5	0.25	−1.133	0.125	−0.420
4	0	0.5	0.125	−0.42001	0.0625	0.01873
5	0.0625	0.01873	0.125	−0.42001	0.09375	−0.2065

Dennis states that B2 is `=EXP(3*A1)-11*A2-0.5`.
(b) Write a correct version of the formula for B2.
The correct formula in A3 is `=IF(F2>0,E2,A2)`.
(c) Write a suitable formula for C3.
(d) Use Fig. 6.2 to find α as accurately as possible and state the maximum possible error.
Liren's spreadsheet for estimates of β, together with further analysis, is:

	x	$f(x)$	difference	ratio
2	0.4	−1.5799		
3	0.6	−1.0504		
4	0.99671	8.4245		
5	0.64398	−0.6809	−0.35273	
6	0.67036	−0.4026	0.026378	−0.0748
7	0.70852	0.08386	0.038164	1.44682
8	0.70194	−0.0075	−0.00658	−0.1724
9	0.70248	−0.0001	0.00054	−0.082
10	0.70249	1.8E−07	8.88E−06	0.01646

The formula in A4 is `=(A2*B3-A3*B2)/(B3-B2)`.
(e) State the method being used.
(f) Explain what the ratios of successive differences $-0.0748, 1.44682, -0.1724, -0.082, 0.01646$ tell you about the order of convergence.
Liren states that $\beta = 0.70249$ correct to 5 decimal places.
(g) Determine whether Liren is correct.

Model solution

(a) Bisection requires the function to have opposite signs at the two endpoints, so that a sign change brackets a root.

$$f(0) = 0.5 > 0, \qquad f(1) = e^3 - 11.5 > 0.$$

Both endpoint values are positive, so $[0, 1]$ does not give a sign-change bracket.

([1])

(b) The spreadsheet must evaluate $e^{3x} - 11x - 0.5$ at the value in A2.

```
=EXP(3*A2)-11*A2-0.5
```

([1])

(c) If $f(x_{\text{new}}) > 0$, the new left endpoint is E2 and the right endpoint remains C2; otherwise the right endpoint becomes E2.

```
=IF(F2>0,C2,E2)
```

([1])

(d) The final displayed bracket for α is $[0.0625, 0.09375]$. Its midpoint is the best estimate and half its width is the maximum error.

$$\alpha \approx \frac{0.0625 + 0.09375}{2} = 0.078125, \qquad \textit{maximum error} = \frac{0.09375 - 0.0625}{2} = 0.015625.$$

$\alpha \approx 0.078125$, with maximum possible error 0.015625.

([2])

(e) The next x-value is formed from the two preceding points and their function values, which is the secant construction.

The secant method.

([1])

(f) For first-order convergence the ratio of successive errors, and hence approximately the ratio of successive differences, tends to a non-zero constant. Here the magnitudes of the ratios decrease overall towards zero, indicating faster-than-first-order convergence.

The decreasing magnitudes of the ratios indicate convergence faster than first order.

([2])

(g) Rounding to 0.70249 *at 5 decimal places is justified only if the root lies in* $[0.702485, 0.702495)$*. Check the signs at these two rounding boundaries.*

$$f(0.702485) \approx -0.0000588, \qquad f(0.702495) \approx 0.0000780.$$

There is a sign change, so the root lies between these bounds.

Yes, $\beta = 0.70249$ *correct to 5 decimal places.*

([2])

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Question 7

Question 7

Fig. 7.1 gives $f(3) = 6.082763$ and $f(3.5) = 4.596194$.
(a) Use the forward difference method to calculate an estimate of the gradient of $f(x)$ at $x = 3$, giving your answer correct to 4 decimal places.
Additional values give the following forward-difference estimates of the gradient at $x = 3$.

h	0.1	0.01	0.001	0.0001	0.00001
estimate	-2.339165	-2.230883	-2.220532	-2.219501	-2.219398
difference		0.1082815	0.010352	0.0010307	0.000103
ratio			0.095602	0.099567	0.0999568

Tommy states that the differences are decreasing so rapidly that the order of convergence is much faster than first order.
(b) Explain whether or not Tommy is correct.
(c) Use extrapolation to determine the gradient at $x = 3$ as accurately as possible, justifying the precision quoted.
(d) Calculate an estimate of the absolute error when $f(3)$ is used as an approximation to $f(3.02)$.

Model solution

(a) A forward difference uses the point at $x = 3$ and the next supplied point, $x = 3.5$.

$$f'(3) \approx \frac{f(3.5) - f(3)}{3.5 - 3} = \frac{4.596194 - 6.082763}{0.5} = -2.973138\ldots$$

$$-2.9731$$

([2])

(b) The differences alone are not enough to identify the order. Their successive ratios approach 0.1, a non-zero constant, which is the signature of first-order convergence as h is divided by 10.

Tommy is not correct: the sequence has first-order convergence.

([2])

(c) Let the last estimate have error e. First-order convergence with ratio $r \approx 0.1$ means the last difference is approximately $(1-r)e$. Hence add $r/(1-r)$ times the final difference to extrapolate to zero step size.

$$f'(3) \approx -2.219398 + 0.000103\left(\frac{0.1}{1 - 0.1}\right) = -2.21938656\ldots$$

The observed ratio 0.0999568 is extremely close to 0.1, supporting the extrapolation. The value is secure to four decimal places, while the fifth decimal is plausible.

$f'(3) \approx -2.2194$ *(secure to 4 d.p.; -2.21939 is probable).*

([5])

(d) For a small increment Δx, the linear approximation gives $f(3.02) - f(3) \approx f'(3)\Delta x$. The question asks for absolute error, so take the magnitude.

$$|f(3.02) - f(3)| \approx |-2.21938656 \times 0.02| = 0.0443877\ldots$$

Estimated absolute error ≈ 0.044388.

([2])

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