Maths with Matt

Physics and maths resources

OCR MEI A-Level Further Mathematics

H645/Y436 - Further Pure with Technology

Sample Assessment Material

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the supplied official OCR mark scheme. The original paper carries 60 marks. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

A family of curves has polar equation

$$r = \cos^n\left(\frac{\theta}{n}\right), \qquad 0 \leq \theta < n\pi,$$

where n is a positive even integer.

(i) (A) Sketch the curve for the cases $n = 2$ and $n = 4$. [2]

(B) State two points which lie on every curve in the family. [1]

(C) State one other feature common to all the curves. [1]

(ii) (A) Write down an integral for the length of the curve for the case $n = 4$. [2]

(B) Evaluate the integral. [2]

(iii) (A) Using $t = \theta$ as the parameter, find a parametric form of the equation of the family of curves. [1]

(B) Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin t \sin\left(\frac{t}{n}\right) - \cos t \cos\left(\frac{t}{n}\right)}{\sin t \cos\left(\frac{t}{n}\right) + \cos t \sin\left(\frac{t}{n}\right)}.$$

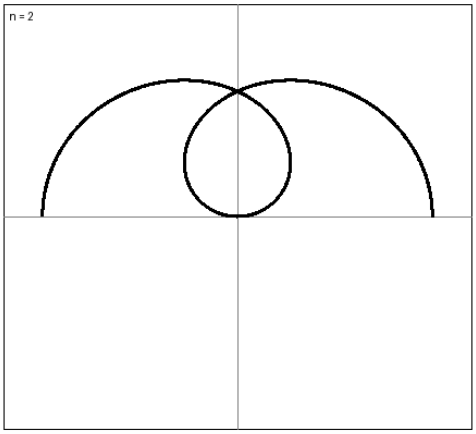
[2]

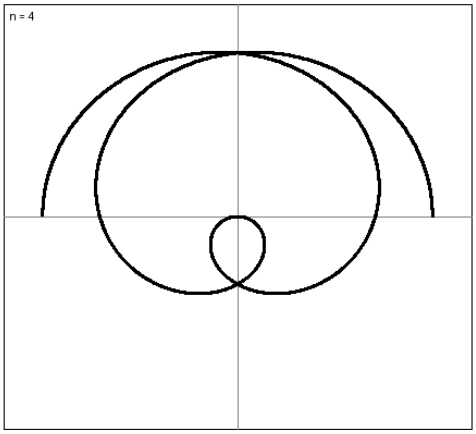
(iv) Hence show that there are $n + 1$ points where the tangent to the curve is parallel to the y-axis. [6]

(v) By referring to appropriate sketches, show that the result in part (iv) is true in the case $n = 4$. [2]

Model solution

(i)(A) A polar sketch is most easily made by letting the given parameter run through its stated interval and plotting $x = r\cos\theta$, $y = r\sin\theta$. The two resulting curves are shown below.





([2])

(i)(B) At $\theta = 0$, $r = 1$, giving $(1, 0)$. At $\theta = \frac{n\pi}{2}$, the cosine factor is zero, so $r = 0$, giving the pole.

$(0, 0)$ *and* $(1, 0)$.

([1])

(i)(C) Since $0 \leq \cos^n(t/n) \leq 1$ *for even* n*, every curve is bounded. Also, replacing* θ *by* $n\pi - \theta$ *gives the reflected point across the* x*-axis, so the curves are symmetric about the* x*-axis. Either observation supplies a common feature.*

All the curves are bounded (and are symmetric about the x-axis).

([1])

(ii)(A) For a polar curve, the arc-length formula is $L = \int \sqrt{r^2 + (\mathrm{d}r/\mathrm{d}\theta)^2}\,\mathrm{d}\theta$*. Here* $r = \cos^4(\theta/4)$*, and the full specified range is* $0 \leq \theta \leq 4\pi$*.*

$$L = \int_0^{4\pi} \sqrt{\cos^8\left(\frac{\theta}{4}\right) + \left(-\cos^3\left(\frac{\theta}{4}\right)\sin\left(\frac{\theta}{4}\right)\right)^2}\,\mathrm{d}\theta.$$

([2])

(ii)(B) The expression under the square root has a common factor $\cos^6(\theta/4)$*. Its square root is* $|\cos^3(\theta/4)|$*, so the absolute value is important because the interval includes values for which the cosine is negative. Put* $u = \theta/4$*.*

$$\begin{aligned} L &= \int_0^{4\pi} \left|\cos^3\left(\frac{\theta}{4}\right)\right| \mathrm{d}\theta = 4\int_0^{\pi} |\cos^3 u|\,\mathrm{d}u \\ &= 8\int_0^{\pi/2} \cos^3 u\,\mathrm{d}u = 8\left[\sin u - \frac{\sin^3 u}{3}\right]_0^{\pi/2} = \frac{16}{3}. \end{aligned}$$

$$L = \frac{16}{3}.$$

([2])

(iii)(A) Convert from polar to Cartesian coordinates, using $t = \theta$*.*

$$x = \cos^n\left(\frac{t}{n}\right)\cos t, \qquad y = \cos^n\left(\frac{t}{n}\right)\sin t, \qquad 0 \leq t < n\pi.$$

$$x = \cos^n(t/n)\cos t, \quad y = \cos^n(t/n)\sin t.$$

([1])

(iii)(B) Differentiate both parametric coordinates with respect to t*, then use* $\mathrm{d}y/\mathrm{d}x = (\mathrm{d}y/\mathrm{d}t)/(\mathrm{d}x/\mathrm{d}t)$*. The common factor* $\cos^{n-1}(t/n)$ *cancels.*

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= -\cos^{n-1}\left(\frac{t}{n}\right)\left(\sin t\cos\left(\frac{t}{n}\right) + \cos t\sin\left(\frac{t}{n}\right)\right), \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \cos^{n-1}\left(\frac{t}{n}\right)\left(\cos t\cos\left(\frac{t}{n}\right) - \sin t\sin\left(\frac{t}{n}\right)\right). \end{aligned}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin t\sin\left(\frac{t}{n}\right) - \cos t\cos\left(\frac{t}{n}\right)}{\sin t\cos\left(\frac{t}{n}\right) + \cos t\sin\left(\frac{t}{n}\right)}.$$

([2])

(iv) A tangent is parallel to the y-axis when $\mathrm{d}x/\mathrm{d}t = 0$, provided $\mathrm{d}y/\mathrm{d}t \neq 0$. From the denominator in the expression just obtained,

$$\sin t \cos\left(\frac{t}{n}\right) + \cos t \sin\left(\frac{t}{n}\right) = 0.$$

$$\sin\left(\frac{(n+1)t}{n}\right) = 0 \quad \Longrightarrow \quad t = \frac{jn\pi}{n+1}, \qquad j = 0, 1, \ldots, n.$$

These are $n+1$ distinct values in $0 \leq t < n\pi$. At each such value the numerator is

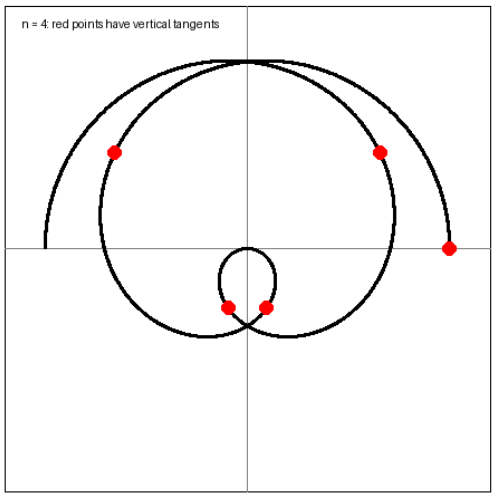
$$-\cos\left(\frac{(n+1)t}{n}\right) = -\cos(j\pi) = (-1)^{j+1} \neq 0.$$

Thus $\mathrm{d}y/\mathrm{d}t \neq 0$, so every one of these parameter values really gives a vertical tangent.

There are exactly $n+1$ points with tangent parallel to the y-axis.

([6])

(v) With $n = 4$, the parameter values are $t = 0, \frac{4\pi}{5}, \frac{8\pi}{5}, \frac{12\pi}{5}, \frac{16\pi}{5}$. The sketch marks the five corresponding points; zooming near the pole makes the two close vertical tangencies clear.



$5 = 4 + 1$ vertical tangencies are visible.

([2])

Question 2

Question 2

(i) (A) Create a program to find all the solutions to $x^2 \equiv -1 \pmod{p}$, where $0 \leq x < p$. Write out your program in full. [5]
(B) Use the program to find the solutions to $x^2 \equiv -1 \pmod{p}$ for the primes $p = 809$, $p = 811$ and $p = 444\,001$. [3]
(ii) State Wilson's Theorem. [1]
(iii) The following argument shows that $(4k)! \equiv ((2k)!)^2 \pmod{p}$ for the case $p = 4k + 1$:

$$\begin{aligned}(4k)! &\equiv 1 \cdot 2 \cdot 3 \cdots (2k-1) \cdot 2k \\ &\quad \cdot (2k+1) \cdot (2k+2) \cdots (4k-1) \cdot 4k \pmod{p} \qquad (1) \\ &\equiv 1 \cdot 2 \cdot 3 \cdots (2k-1) \cdot 2k \\ &\quad \cdot (-2k) \cdot (-(2k-1)) \cdots (-2) \cdot (-1) \pmod{p} \qquad (2) \\ &\equiv ((2k)!)^2 \pmod{p}. \qquad (3)\end{aligned}$$

(A) Explain why $(2k + 2)$ can be written as $-(2k - 1)$ in line (2). [1]
(B) Explain how line (3) has been obtained. [2]
(C) Explain why, if p is a prime of the form $p = 4k + 1$, then $x^2 \equiv -1 \pmod{p}$ will have at least one solution. [1]
(D) Hence find a solution of $x^2 \equiv -1 \pmod{29}$. [2]
(iv) (A) Create a program that will find all the positive integers n, where $n < 1000$, such that

$$(n-1)! \equiv -1 \pmod{n^2}.$$

Write out your program in full. [3]
(B) State the values of n obtained. [2]
(C) A Wilson prime is a prime p such that $(p - 1)! \equiv -1 \pmod{p^2}$. Write down all the Wilson primes p where $p < 1000$. [1]

Model solution

(i)(A) There are only p permitted residues. A direct exhaustive search is therefore appropriate: test every integer from 0 to $p-1$, using the remainder operator so that the comparison is made modulo p.

```
def minus_one_square_solutions(p):
    solutions = []
    for x in range(p):
        if (x * x) % p == p - 1:
            solutions.append(x)
    return solutions
```

([5])

(i)(B) Running the program with each stated prime gives the following complete lists.

p	*solutions* x *with* $0 \leq x < p$
809	$318,\ 491$
811	*none*
$444\,001$	$184\,229,\ 259\,772$

$$p = 809: \ 318,491; \quad p = 811: \textbf{ no solutions;}$$
$$p = 444\,001: \ 184229, 259772.$$

([3])

(ii) The theorem characterises primes using the factorial of the preceding integer.

$$p \textbf{ is prime if and only if } (p-1)! \equiv -1 \pmod{p}.$$

([1])

(iii)(A) Since $p = 4k+1$*,*

$$(2k+2) + (2k-1) = 4k+1 = p.$$

Therefore the two numbers are negatives of one another modulo p*.*

$$2k+2 \equiv -(2k-1) \pmod{p}.$$

([1])

(iii)(B) Each term in the second block of line (2) is the negative of the matching term from $2k$ *down to* 1*. There are* $2k$ *such terms, an even number, so their signs multiply to* $+1$*.*

$$(-2k)(-(2k-1))\cdots(-1) = (-1)^{2k}(2k)! = (2k)!.$$

The first block is also $(2k)!$*, giving the claimed square.*

$$(4k)! \equiv (2k)!(2k)! = ((2k)!)^2 \pmod{p}.$$

([2])

(iii)(C) Wilson's Theorem gives $(4k)! \equiv -1 \pmod{p}$*, because* $p = 4k+1$ *is prime. Combining this with line (3) shows that the displayed congruence has a square root of* -1*.*

$$x = (2k)! \textbf{ is a solution of } x^2 \equiv -1 \pmod{p}.$$

([1])

(iii)(D) Here $29 = 4(7)+1$*, so the result just proved says to use* $x = 14!$ *modulo* 29*. Computing the remainder gives*

$$14! \equiv 12 \pmod{29}, \qquad 12^2 = 144 \equiv -1 \pmod{29}.$$

$$x = 12 \textbf{ is a solution.}$$

([2])

(iv)(A) The condition is checked modulo n^2*, so a loop over all positive* $n < 1000$ *is sufficient. This program computes each factorial exactly, then compares its remainder with the remainder of* -1*.*

```
import math

def wilson_condition_values():
    values = []
```

```
    for n in range(1, 1000):
        if math.factorial(n - 1) % (n * n) == n * n - 1:
            values.append(n)
    return values
```

([3])

(iv)(B) Executing this exhaustive search produces four values and no others in the required range.

$n = 1,\ 5,\ 13,\ 563$, ***with no other values below*** 1000.

([2])

(iv)(C) A Wilson prime must be prime, so remove $n = 1$ from the list in part (B). The remaining listed values are prime.

$5,\ 13,\ 563$.

([1])

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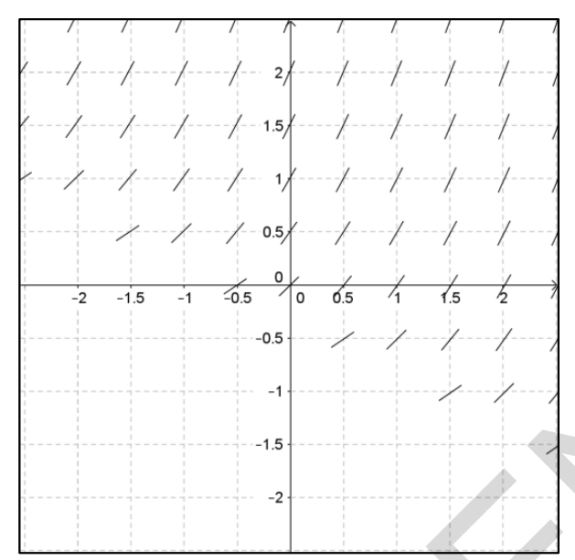
Question 3

Question 3

This question explores the family of differential equations

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{1 + ax + 2y}$$

for various values of the parameter a. Fig. 3 shows the tangent field in the case $a = 1$.



(i) (A) Sketch the tangent field in the case $a = -2$. [2]
(B) Explain why the tangent field is not defined for the whole coordinate plane. [1]
(C) Give an inequality which describes the region in which the tangent field is defined. [1]
(D) Find a value of a such that the region for which the tangent field is defined includes the entire x-axis. [1]
(ii) (A) For the case $a = 1$, with $y = 1$ when $x = 0$, construct a spreadsheet for the Runge–Kutta method of order 2 with formulae

$$k_1 = hf(x_n, y_n), \qquad k_2 = hf(x_n + h, y_n + k_1), \qquad y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2),$$

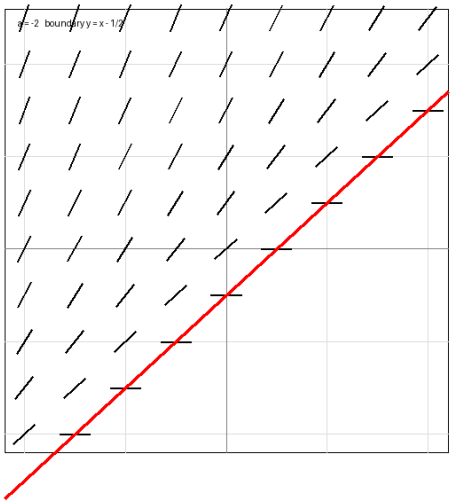
where $f(x, y) = \mathrm{d}y/\mathrm{d}x$. State the formulae used. [3]
(B) Use the spreadsheet to obtain y, correct to 4 decimal places, when $x = 1$, for $h = 0.1$ and $h = 0.05$. [2]
(iii) (A) For the case $a = 0$, find the analytical solution that passes through $(0, 1)$. [1]
(B) Verify that the solution in part (iii)(A) is a solution to the differential equation. [2]
(C) Use the solution in part (iii)(A) to find y, correct to 4 decimal places, when $x = 1$. [1]
(iv) (A) Verify that

$$y = -\frac{a}{2}x + \frac{a^2}{8} - \frac{1}{2}$$

is a solution for all cases when $a \leq 0$. [2]
(B) Show that this is the only straight-line solution in these cases. [4]

Model solution

(i)(A) Substitute $a = -2$, so each short segment has slope $\sqrt{1-2x+2y}$. Segments are drawn only where this is real; the red boundary is where the slope is zero.



([2])

(i)(B) A real tangent field requires the expression inside the square root to be non-negative. At points where it is negative, the proposed gradient is not real.

It is not defined where $1 + ax + 2y < 0$, because the square root is not real.

([1])

(i)(C) Rearranging the non-negative-radicand condition gives the required half-plane.

$$1 + ax + 2y \geq 0 \quad \Longleftrightarrow \quad y \geq -\frac{1}{2} - \frac{a}{2}x.$$

$$y \geq -\frac{1}{2} - \frac{a}{2}x.$$

([1])

(i)(D) On the whole x-axis, $y = 0$, the radicand is $1 + ax$. For this to be non-negative for every real x, its coefficient of x must be zero.

$$a = 0.$$

([1])

(ii)(A) Here $f(x, y) = \sqrt{1 + x + 2y}$. One convenient spreadsheet layout uses column A for h, B for x_n, C for y_n, D for k_1, and E for k_2. The following first-row formulae, filled down, implement the stated method.

```
A2: 0.1
B2: 0
C2: 1
D2: =$A$2*SQRT(1+B2+2*C2)
E2: =$A$2*SQRT(1+(B2+$A$2)+2*(C2+D2))
```

```
B3: =B2+$A$2
C3: =C2+0.5*(D2+E2)
```

For $h = 0.05$, replace the entry in A2 by 0.05 and fill down until $x = 1$. ***([3])***

(ii)(B) Iterating the displayed Runge--Kutta formula to $x = 1$ gives the following values.

h	y **at** $x = 1$
0.1	3.3488
0.05	3.3500

$$y = 3.3488 \textbf{ for } h = 0.1, \textbf{ and } y = 3.3500 \textbf{ for } h = 0.05.$$

([2])

(iii)(A) With $a = 0$, separate variables. The positive square root is appropriate because it is the differential equation's specified right-hand side.

$$\frac{\mathrm{d}y}{\sqrt{1+2y}} = \mathrm{d}x \quad \Longrightarrow \quad \sqrt{1+2y} = x + C.$$

At $(0, 1)$, $C = \sqrt{3}$. Squaring and solving for y gives

$$y = \frac{1}{2}(x+\sqrt{3})^2 - \frac{1}{2} = \frac{1}{2}x^2 + \sqrt{3}x + 1.$$

$$y = \frac{1}{2}x^2 + \sqrt{3}x + 1.$$

([1])

(iii)(B) Differentiate the proposed solution, then compare it directly with the required right-hand side.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x + \sqrt{3}.$$

$$\sqrt{1+2y} = \sqrt{1 + x^2 + 2\sqrt{3}x + 2} = \sqrt{(x+\sqrt{3})^2} = x + \sqrt{3}.$$

On the solution interval from $x = 0$, $x + \sqrt{3} > 0$, so the final equality uses the positive square root. Thus the differential equation is satisfied. ***([2])***

(iii)(C) Substitute $x = 1$ into the exact solution.

$$y = \frac{1}{2} + \sqrt{3} + 1 = 3.232050\ldots$$

$$y = 3.2321.$$

([1])

(iv)(A) Differentiate the proposed straight line; its gradient is $-a/2$. Substitution into the radicand shows why the restriction $a \leq 0$ selects this sign.

$$\sqrt{1+ax+2y} = \sqrt{1 + ax + 2\left(-\frac{a}{2}x + \frac{a^2}{8} - \frac{1}{2}\right)}$$

$$= \sqrt{\frac{a^2}{4}} = -\frac{a}{2}$$

Since $\mathrm{d}y/\mathrm{d}x = -a/2$, *this equals the required derivative. The equality* $\sqrt{a^2/4} = -a/2$ *uses* $a \leq 0$.

$$y = -\frac{a}{2}x + \frac{a^2}{8} - \frac{1}{2}$$ *is the verified straight-line solution.*

([2])

(iv)(B) Let any straight-line solution be $y = mx + c$. *Its derivative is the constant* m, *so after substitution the radicand must be independent of* x.

$$m = \sqrt{1 + ax + 2(mx + c)}.$$

$$m^2 = 1 + (a + 2m)x + 2c.$$

The coefficient of x *is zero, hence*

$$a + 2m = 0 \quad \Longrightarrow \quad m = -\frac{a}{2}.$$

Equating constant terms then determines the intercept uniquely:

$$m^2 = 1 + 2c \quad \Longrightarrow \quad \frac{a^2}{4} = 1 + 2c \quad \Longrightarrow \quad c = \frac{a^2}{8} - \frac{1}{2}.$$

For $a \leq 0$, $m = -a/2 \geq 0$, *so it has the non-negative value required by the square root. No other slope or intercept is possible.*

The only straight-line solution for $a \leq 0$ *is* $y = -\frac{a}{2}x + \frac{a^2}{8} - \frac{1}{2}$.

([4])