Maths with Matt

Physics and maths resources

OCR MEI A Level Further Mathematics

H645/Y421 - Mechanics Major

Tuesday 19 October 2021

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the OCR MEI 2021 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

Question wording is reproduced from the user-provided 2021 paper for study/reference. OCR retains copyright in the original examination material.

Contents

Question 1 2

Question 2 3

Question 3 4

Question 4 5

Question 5 6

Question 6 7

Question 7 8

Question 8 9

Question 9 10

Question 10 11

Question 11 12

Question 12 14

www.mathswithmatt.co.uk

Question 1

Question 1

A ball of mass 0.25 kg falls from rest and strikes a horizontal floor at $5\ \text{m s}^{-1}$. It rebounds at $4.2\ \text{m s}^{-1}$. Its contact time is 0.02 s, and the normal contact force is constant. Determine its magnitude. **(3)**

Model solution

Take upwards as positive. During the very short impact, use impulse = change in momentum; the velocity changes from -5 *to* $+4.2$*.*

$$J = 0.25\{4.2 - (-5)\} = 2.3 \text{ N s}.$$

The constant contact force supplies this impulse over 0.02 *s.*

$$F(0.02) = 2.3, \qquad F = 115.$$

The normal contact force is 115 *N.*

www.mathswithmatt.co.uk

Question 2

Question 2

Particles of masses $3m, 5m, 2m$ are at $A(1,2), B(2,-2), C(5,3)$, respectively. Determine the centre of mass. **(4)**

Model solution

For a system of particles, each coordinate is a mass-weighted mean. The total mass is $10m$.

$$10m\bar{x} = 3m(1) + 5m(2) + 2m(5) = 23m, \qquad \bar{x} = \frac{23}{10} = 2.3,$$

$$10m\bar{y} = 3m(2) + 5m(-2) + 2m(3) = 2m, \qquad \bar{y} = \frac{2}{10} = 0.2.$$

The centre of mass is $(2.3, 0.2)$.

www.mathswithmatt.co.uk

Question 3

Question 3

A light elastic spring has natural length 0.3 m. A 4 kg mass gives an equilibrium extension of 0.02 m. (a) Determine its modulus. (b) A student predicts an extension of 0.1 m for a 20 kg mass. Suggest why this may not be so. **(3+1)**

Model solution

(a) At equilibrium the upward elastic tension balances the weight. Hooke's law $T = \lambda x/l$ *is appropriate while the spring remains elastic.*

$$T = 4g, \qquad \frac{\lambda(0.02)}{0.3} = 4g.$$

$$\lambda = 60g = 588 \text{ N}.$$

$$\lambda = 588 \text{ N}.$$

(b) The calculation assumes Hooke's law still holds at the larger extension.

The spring may be stretched beyond its elastic limit, so its extension need not remain proportional to the load.

www.maths

Question 4

Question 4

The region bounded by $y = 4 - x^2$, $y = 3\sqrt{x}$, and the y-axis is a uniform lamina; the curves meet at $(1, 3)$. Determine the exact x-coordinate of its centre of mass. **(6)**

Model solution

Use vertical strips, since both curves are given as y-functions of x. On $0 \leqslant x \leqslant 1$, $4 - x^2$ is the upper curve and $3\sqrt{x}$ the lower curve.

$$A = \int_0^1 \left((4 - x^2) - 3\sqrt{x}\right) dx = \left[4x - \frac{x^3}{3} - 2x^{3/2}\right]_0^1 = 4 - \frac{1}{3} - 2 = \frac{5}{3}.$$

The first moment about the y-axis is obtained by multiplying each strip's area by its distance x from that axis.

$$A\bar{x} = \int_0^1 x\left((4 - x^2) - 3\sqrt{x}\right) dx = \left[2x^2 - \frac{x^4}{4} - \frac{6}{5}x^{5/2}\right]_0^1 = 2 - \frac{1}{4} - \frac{6}{5} = \frac{11}{20}.$$

$$\bar{x} = \frac{11/20}{5/3} = \frac{33}{100}.$$

The exact x-coordinate is $\frac{33}{100}$.

Question 5

Question 5

Smooth spheres A and B, of masses 2 kg and 4 kg, collide on a horizontal surface. Before impact A travels at 6 m s^{-1} along the line of centres and B travels at 2.5 m s^{-1} perpendicular to it. After impact, B's direction makes 45° with the line of centres. Determine the coefficient of restitution. **(6)**

Model solution

Only components along the line of centres are changed by the smooth impact. Take A's original direction as positive. Since B leaves at $45°$*, its along-line component is* 2.5 *m* s^{-1}*, because its perpendicular component remains* 2.5 *m* s^{-1}*.*

$$2(6) + 4(0) = 2v_A + 4(2.5),$$

$$v_A = 1.$$

Now apply Newton's law of restitution along the line of centres: separation speed is e *times approach speed.*

$$v_A - 2.5 = -e(6 - 0), \qquad 1 - 2.5 = -6e, \qquad e = 0.25.$$

The coefficient of restitution is $\frac{1}{4}$.

Maths with Matt www.mathswithmatt.co.uk

Question 6

Question 6

Gravitational attraction is $F = Gm_1m_2/d^2$. (a) Give dimensions of force. (b) Give dimensions of G. (c) Convert $G = 6.67 \times 10^{-11}\ kg^{-1}m^3s^{-2}$ to pounds, feet and seconds, using $1\ lb = 0.454\ kg$, $1\ ft = 0.305\ m$. (d) Show $v = \sqrt{kGM/r}$, with dimensionless k, is dimensionally consistent. (e) Use Earth data $r = 6371000\ m, M = 5.97 \times 10^{24}\ kg, v = 11186\ m\ s^{-1}$ to find Mars's escape velocity for $r = 3389500\ m, M = 6.39 \times 10^{23}\ kg$. **(1+1+2+3+4)**

Model solution

(a) Force is mass times acceleration.

$$[F] = MLT^{-2}.$$

(b) Rearranging the gravitational law gives $[G] = [F][d]^2/[m]^2$.

$$[G] = (MLT^{-2})L^2M^{-2} = M^{-1}L^3T^{-2}.$$

$$[G] = M^{-1}L^3T^{-2}.$$

(c) Convert each base unit in $kg^{-1}m^3s^{-2}$*: one SI value has the numerical conversion factor* $0.454/(0.305)^3$.

$$G = 6.67 \times 10^{-11}\left(\frac{0.454}{(0.305)^3}\right) = 1.07 \times 10^{-9}\ lb^{-1}ft^3s^{-2}.$$

$$G = 1.07 \times 10^{-9}\ lb^{-1}ft^3s^{-2}.$$

(d) As k *is dimensionless, calculate the dimensions inside the square root.*

$$\left[\frac{kGM}{r}\right] = \frac{(M^{-1}L^3T^{-2})(M)}{L} = L^2T^{-2}.$$

$$\left[\sqrt{\frac{kGM}{r}}\right] = LT^{-1} = [v].$$

The formula is dimensionally consistent.

(e) First use Earth, where the escape velocity is known, to determine the dimensionless constant k.

$$11186^2 = \frac{k(6.67 \times 10^{-11})(5.97 \times 10^{24})}{6371000}, \qquad k \approx 2.$$

$$v = \sqrt{\frac{2(6.67 \times 10^{-11})(6.39 \times 10^{23})}{3389500}} = 5015\ m\ s^{-1}.$$

Mars's escape velocity is $5.02 \times 10^3\ m\ s^{-1}$ *(*$5015\ m\ s^{-1}$*).*

Question 7

Question 7

A box of mass m is raised vertically by an engine of constant power kmg W. It starts from rest; after rising x metres its speed is v. (a) Show $v^2\,dv/dx = (k-v)g$. (b) Verify $gx = k^2\ln(k/(k-v)) - kv - \frac{1}{2}v^2$. (c) Show the time to reach speed V is $\dfrac{k}{g}\ln\left(\dfrac{k}{k-V}\right) - \dfrac{V}{g}$. **(3+5+4)**

Model solution

(a) Power equals force times speed, so the upward driving force is kmg/v. Take upwards as positive and use $a = v\,dv/dx$.

$$\frac{kmg}{v} - mg = m\left(v\frac{dv}{dx}\right).$$

$$\frac{kg}{v} - g = v\frac{dv}{dx} \quad\Longrightarrow\quad v^2\frac{dv}{dx} = (k-v)g.$$

$$v^2\frac{dv}{dx} = (k-v)g.$$

(b) The given expression has the correct initial value: at $x = 0, v = 0$, its right-hand side is 0. Differentiate it implicitly to verify the governing differential equation.

$$g\frac{dx}{dv} = \frac{k^2}{k-v} - k - v = \frac{k^2 - k(k-v) - v(k-v)}{k-v} = \frac{v^2}{k-v}.$$

$$g\frac{dx}{dv} = \frac{v^2}{k-v} \quad\Longrightarrow\quad v^2\frac{dv}{dx} = (k-v)g.$$

This is exactly the result from part (a), with the initial condition also satisfied.

The stated relation is verified.

(c) Use work--energy: engine work equals the gain in kinetic plus gravitational potential energy. Then use part (b) to eliminate x.

$$kmgt = \frac{1}{2}mV^2 + mgx.$$

$$kgt = \frac{1}{2}V^2 + k^2\ln\left(\frac{k}{k-V}\right) - kV - \frac{1}{2}V^2,$$

$$t = \frac{k}{g}\ln\left(\frac{k}{k-V}\right) - \frac{V}{g}.$$

$$t = \frac{k}{g}\ln\left(\frac{k}{k-V}\right) - \frac{V}{g}.$$

Question 8

Question 8

A uniform hollow capsule comprises a cylinder of radius r, length $2h$, and two hemispherical ends. It is in limiting equilibrium between a rough horizontal floor at C and a rough vertical wall at D. Its weight W acts at the midpoint of the cylinder; AB makes acute angle θ with the downward vertical. The coefficient of friction at both contacts is $1/3$. (a) State the remaining forces. (b) Find the floor normal reaction. (c) Show $\tan\theta > 3/4$. **(1+5+6)**

Model solution

(a) At limiting equilibrium, each contact has a normal reaction and a friction force opposing the impending slip. The capsule tends to slip down the wall and away from it at the floor.

At C: normal reaction R_C upwards and friction F_C towards the wall. At D: normal reaction R_D horizontally away from the wall and friction F_D upwards.

(b) Limiting friction has magnitude μR. Resolve horizontally and vertically, with the named directions from part (a).

$$R_D = F_C, \qquad F_D + R_C = W,$$

$$F_C = \frac{1}{3}R_C, \qquad F_D = \frac{1}{3}R_D.$$

$$\frac{1}{3}\left(\frac{1}{3}R_C\right) + R_C = W \quad \Longrightarrow \quad \frac{10}{9}R_C = W.$$

$$R_C = \frac{9}{10}W.$$

(c) Take moments about D, eliminating both forces at the wall. The perpendicular distances follow from resolving the length $2h$ at angle θ.

$$(r + h\sin\theta)W + (r + 2h\cos\theta)F_C = (r + 2h\sin\theta)R_C.$$

Substitute $F_C = R_D = R_C/3 = 3W/10$ and $R_C = 9W/10$, then cancel W.

$$(r + h\sin\theta) + \frac{3}{10}(r + 2h\cos\theta) = \frac{9}{10}(r + 2h\sin\theta),$$

$$r = h\left(2\sin\theta - \frac{3}{2}\cos\theta\right).$$

A radius is positive, so the bracket must be positive.

$$2\sin\theta - \frac{3}{2}\cos\theta > 0 \quad \Longrightarrow \quad 4\sin\theta > 3\cos\theta \quad \Longrightarrow \quad \tan\theta > \frac{3}{4}.$$

$$\tan\theta > \tfrac{3}{4}.$$

Question 9

Question 9

A ball is projected at 5 m s^{-1}, angle $\theta + \alpha$ above the horizontal, from the foot of a plane inclined at α. It lands back on the plane a distance OR uphill. (a) Show $OR = \dfrac{50 \sin\theta \cos(\theta+\alpha)}{g\cos^2\alpha}$. (b) Find its maximum in terms of g and $\sin\alpha$. (c) If the maximum is 1.8 m, find θ. **(8+3+4)**

Model solution

(a) Resolve axes parallel and perpendicular to the plane. Gravity has components $-g\sin\alpha$ *uphill and* $-g\cos\alpha$ *perpendicular to it.*

$$x = 5t\cos\theta - \frac{1}{2}gt^2\sin\alpha, \qquad y = 5t\sin\theta - \frac{1}{2}gt^2\cos\alpha.$$

At the next contact with the plane, $y = 0$ *and* $t \neq 0$.

$$t = \frac{10\sin\theta}{g\cos\alpha}.$$

$$OR = x = \frac{50\sin\theta\cos\theta}{g\cos\alpha} - \frac{50\sin^2\theta\sin\alpha}{g\cos^2\alpha} = \frac{50\sin\theta(\cos\theta\cos\alpha - \sin\theta\sin\alpha)}{g\cos^2\alpha}.$$

$$OR = \frac{50\sin\theta\cos(\theta+\alpha)}{g\cos^2\alpha}.$$

$$\boxed{OR = \frac{50\sin\theta\cos(\theta+\alpha)}{g\cos^2\alpha}.}$$

(b) Apply the supplied product-to-sum identity so that the variable part is a single sine, whose greatest possible value is 1.

$$\sin\theta\cos(\theta+\alpha) = \frac{1}{2}\{\sin(2\theta+\alpha) - \sin\alpha\}.$$

$$OR_{max} = \frac{25(1-\sin\alpha)}{g\cos^2\alpha} = \frac{25}{g(1+\sin\alpha)}.$$

$$\boxed{OR_{max} = \frac{25}{g(1+\sin\alpha)}.}$$

(c) Equate the maximum range to 1.8*, then use the condition for the sine in part (b) to be* 1.

$$\frac{25}{9.8(1+\sin\alpha)} = 1.8 \implies \sin\alpha = \frac{184}{441}, \qquad \alpha = 24.660\ldots^\circ.$$

$$2\theta + \alpha = 90^\circ, \qquad \theta = 45^\circ - \frac{1}{2}\alpha = 32.670\ldots^\circ.$$

$$\boxed{\theta = 32.7^\circ \text{ (0.570 radians).}}$$

Question 10

Question 10

A bead of mass m is projected from B with speed u along a smooth quarter-circular arc BC, radius a, whose centre is vertically above B. At P, $\angle BOP = \theta$. (a) Find the normal reaction. At C it hits a fixed barrier with coefficient of restitution e, then traverses CB and a horizontal rough length $BA = b$, resisted by constant force F. (b) Show it reaches A if $Fb \leqslant \frac{1}{2}m[e^2u^2 + k(1-e^2)ga]$, finding k. **(7+6)**

Model solution

(a) First use conservation of mechanical energy on the smooth arc. Relative to B*, the rise in height at angle* θ *is* $a(1-\cos\theta)$.

$$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + mga(1-\cos\theta),$$

$$v^2 = u^2 - 2ga(1-\cos\theta).$$

Then resolve towards the centre O*. The inward component of weight is* $-mg\cos\theta$*, so Newton's second law gives*

$$R - mg\cos\theta = \frac{mv^2}{a}.$$

$$R = m\left(\frac{u^2}{a} - 2g + 3g\cos\theta\right).$$

$$R = m\left(\frac{u^2}{a} - 2g + 3g\cos\theta\right).$$

(b) At C*, the bead has risen* a*, so energy gives its pre-impact speed squared as* $u^2 - 2ga$*. The barrier reverses the direction and multiplies the speed by* e.

$$v_C^2 = u^2 - 2ga, \qquad \textit{post-impact speed at } C = e\sqrt{u^2 - 2ga}.$$

On returning down the smooth arc to B*, it gains potential energy* mga*. Thus*

$$v_B^2 = 2ga + e^2(u^2 - 2ga).$$

Across BA*, the resistive force does work* Fb*. It reaches* A *exactly or with speed remaining when* $\frac{1}{2}mv_B^2 \geqslant Fb$.

$$Fb \leqslant \frac{1}{2}m\{2ga + e^2(u^2 - 2ga)\} = \frac{1}{2}m\{e^2u^2 + 2(1-e^2)ga\}.$$

The condition holds with $k = 2$.

Question 11

Question 11

Smooth spheres A and B, of masses 4 kg and 3 kg, move in a smooth horizontal circular groove. Initially B is at rest and A has speed V; coefficient of restitution is e, where $e > \frac{2}{5}$. (a) Find speeds after the first collision. (b) Show the angle traversed by A before their second collision is $\frac{2\pi(4-3e)}{7e}$. (c)(i) Find B's speed after the second collision. (ii) Describe the case $e = 1$. **(6+4+5+1)**

Model solution

(a) At a collision the common tangent to the groove is the line of centres, so use one-dimensional momentum and restitution. Let post-impact speeds be v_A, v_B, in A's original direction.

$$4V = 4v_A + 3v_B, \qquad v_A - v_B = -eV.$$

$$v_A = \frac{V(4-3e)}{7}, \qquad v_B = \frac{4V(1+e)}{7}.$$

$$v_A = \frac{V(4-3e)}{7}, \quad v_B = \frac{4V(1+e)}{7}.$$

(b) B is faster and catches A after gaining one full circumference. If the groove radius is r and A has travelled angle θ, equate their times.

$$\frac{r\theta}{V(4-3e)/7} = \frac{2\pi r + r\theta}{4V(1+e)/7}.$$

$$4(1+e)\theta = (4-3e)(2\pi+\theta), \qquad 7e\theta = 2\pi(4-3e).$$

$$\theta = \frac{2\pi(4-3e)}{7e} \text{ radians.}$$

(c)(i) Let the speeds after the second collision be w_A, w_B. Substitute the part (a) speeds into conservation of momentum and Newton's law.

$$3w_B + 4w_A = 3\left(\frac{4V(1+e)}{7}\right) + 4\left(\frac{V(4-3e)}{7}\right) = 4V,$$

$$w_B - w_A = -e\left(\frac{4V(1+e)}{7} - \frac{V(4-3e)}{7}\right) = -e^2V.$$

$$w_B = \frac{4V(1-e^2)}{7}.$$

$$w_B = \frac{4V(1-e^2)}{7}.$$

(c)(ii) Put $e = 1$ *in these results.*

After the second collision B is at rest and A again moves at speed V*; the original state repeats.*

www.mathswithmatt.co.uk

Question 12

Question 12

A particle of mass m is attached to a light elastic string of natural length l, modulus $12mg$, fixed at O. It is released from rest next to O. (a) Show its next instant of rest is at string length $3l/2$. When taut, let extension be x, and let tautening be $t = 0$. (b) Show $\ddot{x} + \omega^2 x = g$, where $\omega^2 = 12g/l$. (c) With $x = y + g/\omega^2$, show first slackening obeys $\cos \omega t - k \sin \omega t = 1$, where k is a constant to be determined. **(5+3+10)**

Model solution

(a) At the next rest position, let the extension be e. Use energy from the release point, taking the horizontal through O as zero gravitational potential energy. The particle has descended $l + e$, and the elastic energy is $\lambda e^2/(2l)$.

$$-mg(l+e) + \frac{12mg\,e^2}{2l} = 0.$$

$$6e^2 - el - l^2 = 0, \qquad (3e+l)(2e-l) = 0.$$

The physical extension is positive, so $e = l/2$, not $-l/3$.

$$\textit{string length} = l + e = \frac{3}{2}l.$$

P next comes to rest at length $\frac{3}{2}l$.

(b) While taut, take downwards as positive. Weight is positive and the elastic tension is upward. Hooke's law gives $T = 12mgx/l$.

$$mg - T = m\ddot{x}, \qquad mg - \frac{12mgx}{l} = m\ddot{x}.$$

$$\ddot{x} + \frac{12g}{l}x = g.$$

$$\ddot{x} + \omega^2 x = g, \quad \omega^2 = \frac{12g}{l}.$$

(c) The suggested shift removes the constant forcing term: $x = y + g/\omega^2$, so y performs simple harmonic motion.

$$\ddot{y} + \omega^2 y = 0, \qquad y = A\cos \omega t + B \sin \omega t,$$

$$x = A\cos \omega t + B \sin \omega t + \frac{g}{\omega^2}.$$

At tautening, $x = 0$, hence $A = -g/\omega^2$. The particle fell freely through l, so its speed at tautening is found by energy.

$$\frac{1}{2}mv_0^2 = mgl, \qquad v_0 = \sqrt{2gl}.$$

Differentiate the displacement expression and put $t = 0$: $\dot{x}(0) = B\omega = v_0$.

$$B = \frac{\sqrt{2gl}}{\omega},$$

$$x = -\frac{g}{\omega^2}\cos\omega t + \frac{\sqrt{2gl}}{\omega}\sin\omega t + \frac{g}{\omega^2}.$$

The string first becomes slack when $x = 0$ again. Use $\omega^2 = 12g/l$ to simplify.

$$x = \frac{l}{12}\left(1 - \cos\omega t + \sqrt{24}\sin\omega t\right) = 0,$$

$$\cos\omega t - \sqrt{24}\sin\omega t = 1.$$

$\cos\omega t - k\sin\omega t = 1$ *with* $k = \sqrt{24} = 2\sqrt{6}$.