Maths with Matt

Physics and maths resources

OCR MEI A Level Further Mathematics

H645/Y431 - Mechanics Minor

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Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the OCR MEI 2021 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

(a) State the dimensions of force. (b) Verify dimensionally that $F = mv^2/r$. (c) Find the percentage increase in F when v increases by 10% and r decreases by 10%. (d) One trackforce is the force which accelerates 1 ton at 1 mile per hour per second, where 1 ton = 1016 kg and 1 mile = 1609 m. Find 1 Tr in newtons.

Model solution

(a) Use $F = ma$. Acceleration has dimensions LT^{-2}, so multiplying by mass gives the dimensions of force.

$$[F] = M(LT^{-2}) = MLT^{-2}.$$

$$[F] = MLT^{-2}.$$

(1)

(b) To check a formula dimensionally, replace every quantity by its dimensions. The two sides must reduce to the same dimensions.

$$\left[\frac{mv^2}{r}\right] = \frac{M(LT^{-1})^2}{L} = \frac{ML^2T^{-2}}{L} = MLT^{-2} = [F].$$

The formula is dimensionally consistent.

(2)

(c) Since $F \propto v^2/r$, it is most direct to compare the new force with the old force; the mass is unchanged.

$$\frac{F_{new}}{F_{old}} = \frac{(1.10v)^2/(0.90r)}{v^2/r} = \frac{1.10^2}{0.90} = 1.34444\ldots$$

$$\textit{percentage increase} = (1.34444\ldots - 1) \times 100 = 34.4\%.$$

34.4% *increase.*

(2)

(d) Convert the stated acceleration into SI units. A speed of one mile per hour is $1609/3600$ m s^{-1}, so its change each second is the same number in m s^{-2}. Then apply $F = ma$.

$$1\ \textit{Tr} = 1016\left(\frac{1609}{3600}\right)\ \textit{N} = 454.095\ldots\ \textit{N}.$$

$1\ \textit{Tr} = 454\ \textit{N (to 3 s.f.).}$

(2)

Question 2

Question 2

A uniform beam rests in limiting equilibrium with end A on rough horizontal ground and end B against a smooth vertical wall. It makes angle θ with the ground. Its weight is W N, the coefficient of friction at A is μ, and the contact-force magnitudes are 70 N at A and 20 N at B. Find W, μ and θ.

Model solution

The smooth wall can exert only a horizontal normal force, so the 20 N force at B is horizontal. At A, resolve the 70 N resultant contact force into horizontal friction F_A and vertical normal reaction R_A. Limiting equilibrium means $F_A = \mu R_A$.

$$F_A = 20, \qquad R_A = \sqrt{70^2 - 20^2} = 30\sqrt{5}.$$

Horizontal forces balance because the beam is at rest. Vertical forces also balance, so the normal reaction at A equals the beam's weight.

$$W = R_A = 30\sqrt{5}\ \text{N} = 67.1\ \text{N}, \qquad \mu = \frac{F_A}{R_A} = \frac{20}{30\sqrt{5}} = \frac{2\sqrt{5}}{15}.$$

To find the inclination, take moments about A. This removes both unknown components of the contact force at A. If the beam length is $2a$, its weight acts at distance a from A and the wall force acts at distance $2a$.

$$W(a\cos\theta) = 20(2a\sin\theta).$$

$$30\sqrt{5}\cos\theta = 40\sin\theta \quad \Rightarrow \quad \tan\theta = \frac{3\sqrt{5}}{4},$$

$$\theta = 59.2^\circ.$$

$$W = 30\sqrt{5}\ \text{N},\ \mu = \frac{2\sqrt{5}}{15},\ \text{and}\ \theta = 59.2^\circ.$$

(7)

Question 3

Question 3

Crates A and B of masses 40 kg and 25 kg are raised by a winch at constant power P. Initially both move upwards at speed v. When the cable between them breaks, A accelerates and later moves at constant speed $(v+3)$ m s^{-1}. Find v and P.

Model solution

At each stated constant speed the acceleration is zero, so the winch force equals the total weight being raised. Constant power then gives $P = (\textit{winch force})(\textit{speed})$.

$$P = (40+25)gv = 65gv$$

After the break, only crate A is raised, and its new constant speed is $v+3$. *The same winch has the same power, hence*

$$P = 40g(v+3).$$

$$65gv = 40g(v+3) \quad \Rightarrow \quad 25v = 120 \quad \Rightarrow \quad v = 4.8 \textit{ m s}^{-1}.$$

$$P = 65(9.8)(4.8) = 3057.6 \textit{ W}.$$

$$v = 4.8 \textit{ m s}^{-1} \textit{ and } P = 3.06 \textit{ kW}.$$

(5)

Question 4

Question 4

A ball of mass m is thrown vertically upwards at 7.2 m s^{-1} from height 1.6 m. (a) Ignoring air resistance, find its maximum height. Thereafter air resistance is a constant force opposite the motion; the actual maximum height is 4.1 m. (b) Show that work against air resistance is $0.568m$ J per metre. (c) Find the speed just before ground impact. (d) After bouncing, it rises to 2.8 m; find the coefficient of restitution. (e) The impulse from the ground at first impact has magnitude 12 N s; find m.

Model solution

(a) With no air resistance, mechanical energy is conserved. The ball rises through a height h, losing all its initial kinetic energy and gaining gravitational potential energy mgh.

$$\frac{1}{2}m(7.2)^2 = mgh \quad \Rightarrow \quad h = \frac{7.2^2}{2(9.8)} = 2.6449\ldots$$

$$\textit{maximum height} = 1.6 + h = 4.24 \textit{ m}.$$

$$4.24 \textit{ m}.$$

(2)

(b) The actual rise is only $4.1 - 1.6 = 2.5$ m. Let W be the work done against air resistance per metre. Energy is spent both against gravity and against air resistance, so use work--energy over this upward journey.

$$\frac{1}{2}m(7.2)^2 = 2.5mg + 2.5W.$$

$$W = \frac{25.92m - 24.5m}{2.5} = 0.568m \textit{ J m}^{-1}.$$

The work against air resistance is $0.568m$ J per metre.

(2)

(c) On the downward journey, gravity does positive work and air resistance does negative work. Start at rest at height 4.1 m and end just before the ground with speed v.

$$mg(4.1) - 4.1(0.568m) = \frac{1}{2}mv^2.$$

$$v^2 = 2(9.8 - 0.568)(4.1) = 75.7024, \qquad v = 8.70 \textit{ m s}^{-1}.$$

The impact speed is 8.70 m s^{-1} downwards.

(3)

(d) Let V be the speed immediately after the bounce. It then rises 2.8 m, so the upward work--energy equation includes both gravity and the downward air resistance.

$$\frac{1}{2}mV^2 = 2.8mg + 2.8(0.568m),$$

$$V = \sqrt{2(9.8 + 0.568)(2.8)} = 7.62 \text{ m s}^{-1}.$$

The ground is stationary, so the coefficient of restitution is the speed of separation divided by the speed of approach.

$$e = \frac{V}{8.7007\ldots} = \frac{7.6197\ldots}{8.7007\ldots} = 0.876.$$

$$e = 0.876.$$

(3)

(e) Take upwards as positive. Before impact the velocity is $-8.7007\ldots$ and after it is $+7.6197\ldots$. The impulse equals the change in momentum.

$$12 = m\big(7.6197\ldots - (-8.7007\ldots)\big),$$

$$m = \frac{12}{16.3204\ldots} = 0.735 \text{ kg}.$$

$$m = 0.735 \text{ kg}.$$

(2)

Question 5

Question 5

A uniform solid L-shaped ornament is 3 cm thick. In the xy-face it consists of a 4×10 rectangle and a 10×3 rectangle attached to its upper right; the axes have origin at the lower left of the 4×10 rectangle. (a) Find its centre of mass. (b) The shaded 4 cm $\times$ 3 cm end face is placed on a sufficiently rough plane inclined at θ; its 4 cm edge is parallel to a line of greatest slope. Find the limiting values of θ for no toppling. (c) With that shaded face on rough horizontal ground, a force P acts at the outer upper edge, at 30° above the horizontal, until toppling is impending but sliding does not occur. Find the minimum μ. (d) State the effect if this angle is less than 30°.

Model solution

(a) Split the solid into two non-overlapping rectangular prisms. Uniform density means their masses are proportional to their volumes. The vertical prism has volume $4 \times 10 \times 3 = 120$ *and centre* $(2, 5, 1.5)$*; the horizontal prism has volume* $10 \times 3 \times 3 = 90$ *and centre* $(9, 8.5, 1.5)$.

$$120 + 90 = 210,$$

$$210 \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = 120 \begin{pmatrix} 2 \\ 5 \end{pmatrix} + 90 \begin{pmatrix} 9 \\ 8.5 \end{pmatrix} = \begin{pmatrix} 1050 \\ 1365 \end{pmatrix}.$$

$$\bar{x} = 5, \qquad \bar{y} = 6.5.$$

Both prisms have the same thickness, so their individual z-coordinates are at the middle of the thickness.

$$\bar{z} = 1.5.$$

The centre of mass is $(5, 6.5, 1.5)$ *cm.*

(4)

(b) Toppling begins when the vertical line through the centre of mass passes through an edge of the contact face. In the cross-section parallel to the greatest slope, the centre is 6.5 *cm above the plane. The two available horizontal clearances to the limiting edges are* 1 *cm and* 5 *cm, so use* $\tan\theta = (\textit{clearance})/(\textit{height})$.

$$\tan\theta_{\min} = \frac{1}{6.5}, \qquad \tan\theta_{\max} = \frac{5}{6.5}.$$

$$\theta_{\min} = 8.75^\circ, \qquad \theta_{\max} = 37.6^\circ.$$

$$8.75^\circ \leqslant \theta \leqslant 37.6^\circ.$$

(3)

(c) There are two possible ways to lose equilibrium: slide or topple. Find the value of P *needed for each, then require the toppling threshold to be no larger than the sliding threshold. The upward component of* P *reduces the normal reaction.*

$$R + P\sin 30^\circ = mg \quad \Rightarrow \quad R = mg - P\sin 30^\circ.$$

At impending sliding, the horizontal component equals limiting friction μR. Hence

$$P_s \cos 30^\circ = \mu\left(mg - P_s \sin 30^\circ\right),$$

$$P_s = \frac{\mu mg}{\cos 30^\circ + \mu \sin 30^\circ}.$$

At impending toppling, take moments about the lower-left contact edge. The weight has clockwise moment $5mg$. The vertical and horizontal components of P have anticlockwise moment arms 14 cm and 10 cm respectively; centimetres cancel from the moment equation.

$$14P_t \sin 30^\circ + 10P_t \cos 30^\circ = 5mg,$$

$$P_t = \frac{5mg}{14 \sin 30^\circ + 10 \cos 30^\circ}.$$

The statement "toppling but does not slide" requires $P_s > P_t$. Since all denominators are positive, substitute and rearrange without changing the inequality direction.

$$\frac{\mu mg}{\cos 30^\circ + \mu \sin 30^\circ} > \frac{5mg}{14 \sin 30^\circ + 10 \cos 30^\circ},$$

$$\mu(14 \sin 30^\circ + 10 \cos 30^\circ) > 5(\cos 30^\circ + \mu \sin 30^\circ),$$

$$\mu(9 \sin 30^\circ + 10 \cos 30^\circ) > 5 \cos 30^\circ,$$

$$\mu > \frac{5}{9 \tan 30^\circ + 10} = \frac{50 - 15\sqrt{3}}{73} = 0.329\ldots$$

$$\mu_{\min} = \frac{50 - 15\sqrt{3}}{73} \approx 0.329.$$

(7)

(d) The expression $\mu_{\min} = 5/(9 \tan \theta + 10)$ applies with the force angle θ. If the angle is smaller, $\tan \theta$ is smaller, so the denominator is smaller.

The minimum coefficient of friction would be larger.

(2)

Question 6

Question 6

(a) Show that a block projected on a rough horizontal surface with speed v travels $v^2/(2\mu g)$ before stopping. A sphere S and block B are on the same surface, with a wall beyond S; $m_B = 8m_S$. The surface under S is smooth and that under B is rough. Sphere S is projected towards stationary B at speed u. The coefficient of restitution for S–B is 0.8, and wall impacts are perfectly elastic. (b) Find the speeds and directions just after the first S–B impact. (c) Given that B stops before each later collision, explain why $x_{n+1}/x_n = 9/25$. (d) If $u = 11.2$ and $\mu = 1/7$, show that the total distance travelled by B cannot exceed 2.8 m.

Model solution

(a) While the block slides, friction has magnitude $\mu R = \mu mg$ and does negative work. Apply work--energy from projection to rest over distance x.

$$0 - \frac{1}{2}mv^2 = -\mu mgx.$$

$$x = \frac{v^2}{2\mu g}.$$

$$\boxed{x = \frac{v^2}{2\mu g}.}$$

(3)

(b) Take the initial direction of S as positive. Write v_S and v_B for the velocities after impact. Conservation of momentum applies during the short collision, and Newton's law of restitution relates relative speeds.

$$m_S u = m_S v_S + 8m_S v_B \quad \Rightarrow \quad u = v_S + 8v_B,$$

$$v_B - v_S = 0.8u.$$

Substitute $v_B = 0.8u + v_S$ into the momentum equation.

$$u = v_S + 8(0.8u + v_S) = 9v_S + 6.4u,$$

$$v_S = -0.6u, \qquad v_B = 0.2u.$$

S has speed $0.6u$ towards the wall; B has speed $0.2u$ away from the wall.

(5)

(c) The perfectly elastic wall impact reverses S without changing its speed. On its return to a stationary B, the collision calculation is a scaled copy of part (b): the speed leaving S is

$3/5$ *of its incoming speed, and the speed given to* B *is therefore also* $3/5$ *of the previous post-impact speed of* B*.*

$$\frac{v_{B,n+1}}{v_{B,n}} = \frac{3}{5}.$$

Part (a) shows that stopping distance is proportional to the square of the initial speed. Therefore

$$\frac{x_{n+1}}{x_n} = \left(\frac{3}{5}\right)^2 = \frac{9}{25}.$$

$$x_{n+1} = \frac{9}{25}x_n.$$

(2)

(d) From part (b), the initial speed of B *after the first impact is* $0.2u = 2.24$ *m s*$^{-1}$*. Use the stopping-distance result, then sum the geometric series of successive distances.*

$$x_1 = \frac{2.24^2}{2(1/7)(9.8)} = 1.792 \text{ m}.$$

$$\text{total distance} = 1.792\left(1 + \frac{9}{25} + \left(\frac{9}{25}\right)^2 + \cdots\right) = \frac{1.792}{1 - 9/25} = 2.8 \text{ m}.$$

This is the infinite-series value, so the finite distance travelled at any stage cannot exceed it.

The total distance travelled by B *cannot exceed* 2.8 *m.*

(3)