Maths with Matt

Physics and maths resources

OCR MEI A Level Further Mathematics

H645/Y433 - Modelling with Algorithms

Wednesday 20 October 2021

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the OCR MEI 2021 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

Thirteen items, in this order, have sizes

$$5, 16, 12, 15, 21, 10, 17, 5, 3, 6, 13, 24, 5.$$

Bins have capacity 45. (a) Apply first fit. (b) Apply a full-bin strategy.

Model solution

(a) First fit means that each item is considered in the stated order and put into the first existing bin with enough spare capacity. It is important not to rearrange the items for this algorithm.

bin	*items placed in it*	*total*
1	5, 16, 12, 10	43
2	15, 21, 5, 3	44
3	17, 6, 13, 5	41
4	24	24

The first-fit packing is $(5, 16, 12, 10)$, $(15, 21, 5, 3)$, $(17, 6, 13, 5)$ *and* (24).

(2)

(b) A full-bin strategy looks for combinations that use the whole capacity, so begin with the large items and complete bins of total 45.

bin	*items*	*total*
1	24, 21	45
2	16, 13, 10, 6	45
3	17, 15, 5, 5, 3	45
4	12, 5	17

One full-bin packing has three full bins and one bin containing 12 *and* 5.

(2)

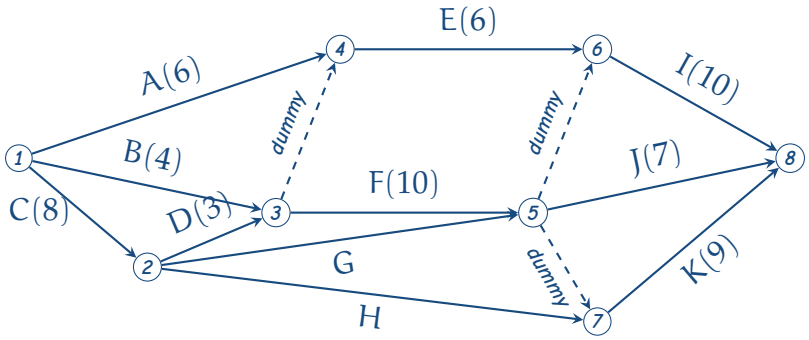
Question 2

Question 2

The project activities are A(6), B(4) and C(8) with no predecessors; D(3) after C; E(6) after A, B, D; F(10) after B, D; G and H after C; I(10) after E, F, G; J(7) after F, G; and K(9) after F, G, H. The only critical activities are C, D, F, I. (a) Draw an activity-on-arc network. (b) Perform forward and backward passes. (c) State the minimum completion time. (d) Find the interfering float of H, given that its independent float is not 0. (e) If x is the duration of G, and the total float of G is at most double that of E, find the possible values of x.

Model solution

(a) The activity-on-arc network is below. Dashed arrows are dummy activities of duration 0.



The network has one start, one finish and exactly three dummy arcs.

(3)

(b) In a forward pass, an event time is the largest of the times arriving there: every predecessor must have finished. In a backward pass, work from the completion time and take the smallest time allowed by outgoing activities. The given critical chain C − D − F − I *gives the key times* 0 + 8 + 3 + 10 + 10 = 31.

event	*description*	*early time*	*late time*
1	*start*	0	0
2	*end of* C; *start of* D, G, H	8	8
3	*end of* B, D; *start of* F	11	11
4	*end of* A; *start of* E	11	15
5	*end of* F, G; *start of* J	21	21
6	*end of* E; *start of* I	21	21
7	*end of* H; *start of* K	21	22
8	*finish*	31	31

In particular, event 4, *the separate start event of* E, *has early time* 11 *and late time* 15. *The equal early and late times on the critical chain confirm* C − D − F − I.

The forward and backward passes give a project duration of 31 *hours, with the critical events at* 0, 8, 11, 21, 31.

(3)

(c) The completion time is the early time at the finish event.

31 *hours.*

(1)

(d) Interfering float is total float minus independent float. Using the start event of H *(early time* 8*), its end event (early time* 21*, late time* 22*), the difference can be found without needing the duration of* H*.*

$$\textit{interfering float} = (22 - 8) - (21 - 8) = 1.$$

1 *hour.*

(1)

(e) Use total float = late time at end − early time at start − duration. This is appropriate because the condition compares total floats.

$$\mathrm{TF(E)} = 21 - 11 - 6 = 4, \qquad \mathrm{TF(G)} = 21 - 8 - x = 13 - x.$$

$$13 - x \leqslant 2(4) \quad \Rightarrow \quad x \geqslant 5.$$

Activity G *must also fit before its event at time* 21*, so its duration is less than* 13 *hours; equality would change the stated critical-activity information.*

$5 \leqslant x < 13$ *hours.*

(3)

Question 3

Question 3

An odd vertex has odd order. (a) Prove facts about odd vertices. For the weighted graph in Fig. 3, find the shortest path from A to F by Dijkstra's algorithm; then apply the route-inspection algorithm. The odd vertices are A, E, G, I, and the total weight is 353. The relevant weighted arcs are $AB = 25$, $AC = 10$, $AD = 20$, $BC = 12$, $BE = 15$, $BG = 10$, $CE = 32$, $CD = 8$, $CG = 27$, $CF = 72$, $DG = 16$, $DH = 18$, $EG = 41$, $GI = 18$, $FI = 17$, $HI = 12$.

Model solution

(a)(i) Count arc endings. This method works for every graph because each arc contributes exactly two endings.

$$\sum \textit{(vertex orders)} = 2 \times \textit{(number of arcs)}.$$

The sum of the vertex orders is even.

(1)

(a)(ii) The orders of the even vertices have an even sum. Since the sum of all orders is even, the sum of the odd vertex orders must also be even. A sum of an odd number of odd integers is odd, so this is impossible.

A graph must have an even number of odd vertices; it cannot have an odd number of them.

(1)

(b) Dijkstra's algorithm is suitable because all arc lengths are non-negative. Label each vertex with the shortest known distance from A, permanently fixing the smallest temporary label at each step.

permanent vertex	*permanent label*	*useful new/improved labels*
A	0	C : 10, D : 20, B : 25
C	10	D : 18, B : 22, E : 42, G : 37, F : 82
D	18	G : 34, H : 36
B	22	E : 37, G : 32
G	32	I : 50
H	36	I : 48
E	37	
I	48	F : 65
F	65	

Tracing the predecessor labels backwards from F *gives* $F \leftarrow I \leftarrow H \leftarrow D \leftarrow C \leftarrow A$.

The shortest path is $A - C - D - H - I - F$, *with length* 65.

(5)

(c) Route inspection first pairs the odd vertices using their shortest connecting paths.

Listing every pair makes it possible to choose the least total pairing in the next step.

odd pair	*shortest path*	*weight*
AE	A – C – B – E	37
AG	A – C – B – G	32
AI	A – C – D – H – I	48
EG	E – B – G	25
EI	E – B – G – I	43
GI	G – I	18

The completed Step 1 table is as shown.

(4)

(d) Pair all four odd vertices exactly once. The three possible pairing totals are $37 + 18$, $32 + 43$ *and* $48 + 25$, *so the first is smallest. Repeating these shortest paths makes every vertex even, allowing a closed route.*

$$\textit{extra weight} = 37 + 18 = 55, \qquad 353 + 55 = 408.$$

Choose AE *(via* A – C – B – E*) and* GI*; the shortest closed route has length* 408*.*

(2)

Question 4

Question 4

Fig. 4.1 is a directed pipe network from source S to sink T. Its arcs (capacity in litres per minute) are

$$\mathrm{SA}:62,\ \mathrm{SB}:71,\ \mathrm{SC}:47,\ \mathrm{AB}:43,\ \mathrm{AD}:22,\ \mathrm{BD}:39,\ \mathrm{BE}:32,\ \mathrm{BF}:43,$$
$$\mathrm{BG}:47,\ \mathrm{CB}:25,\ \mathrm{CF}:39,\ \mathrm{DE}:33,\ \mathrm{DT}:82,\ \mathrm{EG}:43,\ \mathrm{ET}:24,\ \mathrm{FG}:42,\ \mathrm{GT}:67.$$

Cuts α and β have the indicated capacities. Answer the maximum-flow and LP questions, including the case where capacity AD is increased to x.

Model solution

(a)(i) The capacity of a cut is the sum of capacities of all arcs crossing it from the source side to the sink side. For cut α, these are $\mathrm{AD}, \mathrm{AB}, \mathrm{SB}, \mathrm{BG}$.

$$22 + 43 + 71 + 47 = 183.$$

$$\mathrm{cap}(\alpha) = 183 \text{ litres per minute.}$$

(1)

(a)(ii) Apply the same cut rule to β.

$$82 + 33 + 43 + 71 + 25 + 39 = 293.$$

$$\mathrm{cap}(\beta) = 293 \text{ litres per minute.}$$

(1)

(b) Every feasible flow is no larger than every cut capacity. Using only the two given cuts, the tighter upper bound is the smaller one.

The maximum possible flow is at most 183 litres per minute.

(1)

(c) At an intermediate vertex, conservation of flow is required. Here SC is the only inflow to C, while CB and CF are its only outflows.

$$\mathrm{SC} - \mathrm{CB} - \mathrm{CF} = 0 \iff \mathrm{SC} = \mathrm{CB} + \mathrm{CF}.$$

The constraint enforces flow in $=$ flow out at C.

(1)

(d) To maximise flow, use the total flow entering the sink as the objective. Add the missing

conservation equations at B *and* E*, and capacity constraints for the three arcs entering* T*.*

$$\text{Maximise } DT + ET + GT$$
$$SB + AB + CB - BD - BE - BF - BG = 0,$$
$$BE + DE - EG - ET = 0,$$
$$DT \leqslant 82, \qquad ET \leqslant 24, \qquad GT \leqslant 67.$$

These, together with the supplied constraints and capacity bounds, complete the LP.

(3)

(e) Use the displayed solver values and conservation of flow to fill the remaining arc flows. Each is at most its capacity; checking each intermediate vertex confirms that inflow equals outflow.

arc	SA	SB	SC	AB	AD	BD	BE	BF	BG
flow	34	71	47	12	22	39	24	3	25

arc	CB	CF	DE	EG	DT	ET	FG	GT
flow	8	39	0	0	61	24	42	67

$$61 + 24 + 67 = 152.$$

A feasible flow of 152 *litres per minute is given by the table.*

(2)

(f) To prove optimality, find a cut with the same capacity as the feasible flow. Put $\{S, A, B, C, E, F, G\}$ *on the source side and* $\{D, T\}$ *on the sink side.*

$$\text{cap}(\textit{cut}) = AD + BD + ET + GT = 22 + 39 + 24 + 67 = 152.$$

The maximum-flow minimum-cut theorem says that the maximum flow equals the capacity of a minimum cut. Since a flow and a cut both have value 152*, both are optimal.*

152 *litres per minute is the maximum flow.*

(2)

(g) After increasing AD*, the source arc* SA *has unused capacity* $62 - 34 = 28$*, while sink arc* DT *has unused capacity* $82 - 61 = 21$*. These are the limiting residual amounts, so the flow can increase by* 21*.*

$$\textit{new maximum} = 152 + 21 = 173, \qquad x = 22 + 21 = 43.$$

The maximum flow is 173 *litres per minute when* $x = 43$ *litres per minute.*

(2)

Question 5

Question 5

Leo answers exactly 50 questions in 210 minutes. Let x, y, z be the numbers of algebra, trigonometry and calculus questions. Their times are $1, 4, 12$ minutes and their scores are $2, 5, 20$ points respectively. He has $x \leqslant 25$ and $y \geqslant 3z$. Maximise his score.

Model solution

(a) Two-stage simplex is appropriate because the equality "exactly 50" needs an artificial variable to provide an initial basic variable. First express it as both $\leqslant$ and $\geqslant$, then add slack variables to $\leqslant$ constraints and a surplus plus artificial variable to the $\geqslant$ constraint.

$$\begin{aligned} x + y + z + s_1 &= 50, \\ x + y + z - s_2 + a_1 &= 50, \\ x + s_3 &= 25, \\ -y + 3z + s_4 &= 0, \\ x + 4y + 12z + s_5 &= 210. \end{aligned}$$

The scoring objective is $P = 2x + 5y + 20z$. In stage 1 minimise the artificial variable; substituting from the equality row gives the objective row in the tableau.

$$P - 2x - 5y - 20z = 0, \qquad Q = a_1 = 50 - x - y - z + s_2,$$

$$Q + x + y + z - s_2 = 50.$$

	Q	P	x	y	z	s_1	s_2	s_3	s_4	s_5	a_1	*RHS*
Q	1	0	1	1	1	0	−1	0	0	0	0	50
P	0	1	−2	−5	−20	0	0	0	0	0	0	0
	0	0	1	1	1	1	0	0	0	0	0	50
	0	0	1	1	1	0	−1	0	0	0	1	50
	0	0	1	0	0	0	0	1	0	0	0	25
	0	0	0	−1	3	0	0	0	1	0	0	0
	0	0	1	4	12	0	0	0	0	1	0	210

The equations, objective rows and initial tableau above set up the two-stage simplex method.

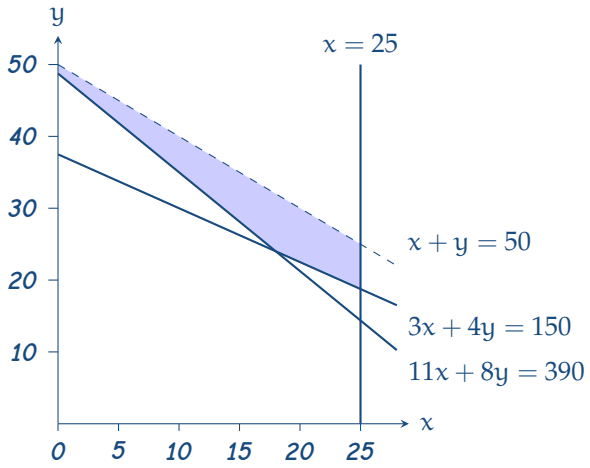
(9)

(b) Eliminate z using $z = 50 - x - y$, because the feasible region is requested in the x, y plane.

$$x \leqslant 25, \qquad -y + 3(50 - x - y) \leqslant 0, \qquad x + 4y + 12(50 - x - y) \leqslant 210,$$

$$\boxed{x \leqslant 25, \qquad 3x + 4y \geqslant 150, \qquad 11x + 8y \geqslant 390.}$$

Also $x \geqslant 0$, $y \geqslant 0$ *and* $z = 50 - x - y \geqslant 0$. *The graph shows the three required boundary lines and the shaded feasible region (also below* $x + y = 50$*).*



The feasible polygon has boundary vertices $(0, 50)$, $(25, 25)$, $(25, 75/4)$, $(18, 24)$ *and* $(0, 195/4)$. *Its relevant corner for the optimisation is the intersection of* $3x + 4y = 150$ *and* $11x + 8y = 390$.

$$3x + 4y = 150, \quad 11x + 8y = 390 \quad \Rightarrow \quad x = 18, \ y = 24.$$

The feasible region is the intersection of the stated half-planes (and $x, y \geqslant 0$, $x + y \leqslant 50$*).*

(5)

(c) Substitute $z = 50 - x - y$ *into the score, so that only* x *and* y *remain.*

$$P = 2x + 5y + 20(50 - x - y) = 1000 - 18x - 15y = 1000 - 3(6x + 5y).$$

Maximising P *is therefore the same as minimising* $6x + 5y$*, since multiplying by the positive constant* 3 *does not change where the minimum occurs.*

The optimal x, y *minimise* $6x + 5y$.

(2)

(d)(i) The minimum objective line meets the feasible region at the intersection found above. Recover z *from the total-number constraint.*

$$x = 18, \qquad y = 24, \qquad z = 50 - 18 - 24 = 8.$$

Leo should answer 18 *algebra,* 24 *trigonometry and* 8 *calculus questions.*

(1)

(d)(ii) Evaluate the original score function, using the numbers of questions of each type.

$$P = 2(18) + 5(24) + 20(8) = 36 + 120 + 160 = 316.$$

316 *points.*

(1)

(e) The model maximises the score available from the questions attempted; it assumes every answer is correct.

Leo may answer one or more questions incorrectly, so he is not guaranteed to score 316 *points.*

(1)

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