Maths with Matt

Physics and maths resources

OCR MEI A Level Further Mathematics

H645/Y434 - Numerical Methods

Friday 22 October 2021

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the OCR MEI 2021 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

(a) (i) Determine the relative error when 1.414214 is used to approximate $\sqrt{2}$, and when 1.414214^2 is used to approximate 2. [3]
(ii) State the relationship between the two relative errors. [1]
(b) A spreadsheet displays 1.414214 in cell B1, where B1 contains `=SQRT(A1)` and A1 $= 2$. Cell C1 contains `=B1^2` and displays 2. Ben obtains $1.414214^2 = 2.000001238$ on his calculator and says C1 is wrong. Explain whether he is correct. [1]

Model solution

(a)(i) Relative error compares the error with the exact value. We use the exact values $\sqrt{2}$ and 2, respectively.

$$\frac{1.414214 - \sqrt{2}}{\sqrt{2}} = 3.09449 \times 10^{-7}, \qquad \frac{1.414214^2 - 2}{2} = 6.18898 \times 10^{-7}.$$

The relative errors are 3.09449×10^{-7} and 6.18898×10^{-7}, respectively.

(3)

(a)(ii) Comparing the two numbers shows the effect of squaring a value close to $\sqrt{2}$.

The second relative error is twice the first.

(1)

(b) A displayed decimal need not be the value used in a spreadsheet calculation: spreadsheets normally retain extra digits internally.

Ben is not correct. B1 can store $\sqrt{2}$ to greater precision than the displayed 1.414214, so squaring its stored value gives 2.

(1)

Question 2

Question 2

For the data

x	1	2	3	4	5
f(x)	−0.65	−0.35	1.77	5.71	11.47

(**a**) complete the difference table; (**b**) explain why a degree-2 interpolating polynomial is appropriate; (**c**) use Newton's forward-difference formula to find that polynomial. [2, 1, 3]

Model solution

(a) Successive first differences are found by subtracting each entry from the next one. Repeating this gives the second differences.

x	f(x)	Δf	$\Delta^2 f$
1	−0.65	0.30	1.82
2	−0.35	2.12	1.82
3	1.77	3.94	1.82
4	5.71	5.76	
5	11.47		

$\Delta f = 0.30, 2.12, 3.94, 5.76$, *and* $\Delta^2 f = 1.82, 1.82, 1.82$.

(2)

(b) Constant second differences are the finite-difference signature of a quadratic, just as a constant second derivative is for a quadratic function.

The second differences are constant, so a polynomial of degree 2 is appropriate.

(1)

(c) The points are equally spaced with step $h = 1$, and the first value is at $x_0 = 1$, so Newton's forward formula uses $u = (x - x_0)/h = x - 1$.

$$P_2(x) = f_0 + u\Delta f_0 + \frac{u(u-1)}{2!}\Delta^2 f_0 = -0.65 + 0.30(x-1) + \frac{1.82(x-1)(x-2)}{2}.$$

$$P_2(x) = 0.91x^2 - 2.43x + 0.87.$$

$$\boxed{P_2(x) = 0.91x^2 - 2.43x + 0.87}.$$

(3)

Question 3

Question 3

False Position is applied using the spreadsheet columns $a, f(a), b, f(b), x_{new}, f(x_{new})$. The formula in cell D5 is `=SINH(C5^2)-C5^3-2`. The estimates include

a	$f(a)$	b	$f(b)$	x_{new}
1.48719	-0.77825	2	17.2899	(next)

Cell C6 is `=IF(H5>0,G5,C5)`. **(a)** State the equation. **(b)** Give the corresponding formula for E6. **(c)** Find G13 and G14. **(d)** Interpret ratios $0.88229, 0.86458, \ldots, 0.78598$. [1,1,3,2]

Model solution

(a) The spreadsheet formula is the function being set equal to zero.

$$\boxed{\sinh(x^2) - x^3 - 2 = 0}.$$

(1)

(b) C6 replaces the lower endpoint when the new function value is positive. The upper endpoint must therefore be replaced when it is negative.

`=IF(H5>0,G5,E5)`

(1)

(c) False Position uses the x-intercept of the chord through $(a, f(a))$ and $(b, f(b))$: $x_{new} = \{af(b) - bf(a)\}/\{f(b) - f(a)\}$.

$$G13 = \frac{1.48719(17.2899) - 2(-0.77825)}{17.2899 - (-0.77825)} = 1.50928\ldots$$

Here $f(1.50928) \approx -0.6111$, so this becomes the new lower endpoint; use it with $b = 2$ for the next chord.

$$G14 = 1.52603\ldots$$

$$G13 \approx 1.50928,\ G14 \approx 1.52603.$$

(3)

(d) The ratio is the next change divided by the preceding change. A ratio tending to a non-zero value indicates first-order behaviour; here it is falling, so the order is slightly higher.

The convergence is slightly faster than first order (between first and second order), but the ratios are close to 1, so convergence is slow.

(2)

Question 4

Question 4

x	4	4.0001	4.001	4.01	4.1
$f(x)$	4	4.0002386	4.0023871	4.0239468	4.2472072

(a) Calculate four estimates of $f'(4)$. **(b)** State $f'(4)$ as accurately as possible without further calculation, and justify the precision. [4,2]

Model solution

(a) All values lie to the right of $x = 4$, so use the forward-difference estimate $[f(4+h) - f(4)]/h$ for each available h.

h	$\dfrac{f(4+h)-f(4)}{h}$
0.1	$\dfrac{4.2472072 - 4}{0.1} = 2.472072$
0.01	$\dfrac{4.0239468 - 4}{0.01} = 2.39468$
0.001	$\dfrac{4.0023871 - 4}{0.001} = 2.3871$
0.0001	$\dfrac{4.0002386 - 4}{0.0001} = 2.386$

The four estimates are $2.472072, 2.39468, 2.3871, 2.386$.

(4)

(b) The two smallest-step estimates agree to two decimal places, whereas their third decimal places differ.

$\boxed{f'(4) \approx 2.39}$ *, secure to 2 decimal places.*

(2)

Question 5

Question 5

Nina rounds each item price to the nearest pound. She buys 48 items, estimates \$92, and the actual total is \$90.23. Kareem chops each price down to a whole number of pounds. He buys 52 items and estimates \$76, whereas the actual total is \$103.24. He later buys n items.
Find or explain: (**a**) Nina's maximum possible error; (**b**) consistency of her result; (**c**) Kareem's expected error for 52 items; (**d**) how his large error can arise; (**e**) his maximum possible error for n items; (**f**) an adaptation giving Nina's expected error. [2,1,2,1,1,2]

Model solution

(a) Rounding to the nearest pound changes one price by at most $\$0.50$. *In the worst case all errors have the same sign.*

$$48 \times \$0.50 = \$24.$$

Maximum possible error: $\$24$.

(2)

(b) The magnitude of the observed error is $\$92 - \$90.23 = \$1.77$, *which must be no larger than the maximum possible error.*

Yes: $\$1.77 < \$24$, *so it is consistent.*

(1)

(c) Chopping loses from 0*p up to* 99*p on an item. Assuming pennies are evenly spread, the expected loss per item is* 49.5*p.*

$$52 \times \$0.495 = \$25.74.$$

The expected error (actual total minus estimate) is $\$25.74$.

(2)

(d) Unlike rounding, chopping is systematically downward, so errors can accumulate.

Many prices could have been just below the next pound, for example $\$1.99$ *or* $\$2.99$, *giving an average loss of about* 52.38*p per item.*

(1)

(e) The largest possible loss per chopped item is 99*p.*

Maximum possible error $= \$0.99n$.

(1)

(f) Nearest-pound rounding has expected error 0, *because upward and downward errors balance. Kareem's chopped estimate is expected to be too small by* $\$0.495n$.

Use adapted estimate = chopped total + $0.495n; its expected error is then 0, the same as Nina's.

(2)

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Question 6

Question 6

The equation $0.5\ln x - x^2 + x + 1 = 0$ has roots α and β, where $0 < \alpha < 1$ and $1 < \beta < 2$.
(a) Use Newton–Raphson from $x_0 = 1$ to obtain β. **(b)** On the given graph, illustrate the step from $x_0 = 1$ to x_1. **(c)** Give a reason to prefer Newton–Raphson to fixed-point iteration for α. **(d)** Starting Newton–Raphson at 0.5 gives $x_1 = -0.40343$, then `#NUM!`; explain. For $g(x) = \sqrt{0.5\ln x + x + 1}$, **(e)** determine the outcome from $x_0 = 0.5$; **(f)** use $x_{n+1} = (1-\lambda)x_n + \lambda g(x_n)$, $\lambda = -0.041$, from 0.5.
[4,2,1,1,2,2]

Model solution

(a) Newton--Raphson is appropriate because the function and its derivative are available. With $f(x) = 0.5\ln x - x^2 + x + 1$, its tangent-gradient is

$$f'(x) = \frac{1}{2x} - 2x + 1.$$

Each tangent's x-intercept gives the next estimate: $x_{n+1} = x_n - f(x_n)/f'(x_n)$.

n	0	1	2	3	4	5
x_n	1	3	2.0791668	1.7783346	1.7360141	1.7351281

$$x_6 = 1.7351277\ldots$$

$\boxed{\beta = 1.735128}$ *(6 d.p.).*

(4)

(b) A Newton--Raphson step is read from a tangent, not from the curve itself. Since $f(1) = 1$, start at $(1, 1)$.

Draw the tangent to the curve at $(1, 1)$; it meets the x-axis at $(3, 0)$. Project this intercept to the axis to show $x_1 = 3$.

(2)

(c) The clue is that a numerical root is required to six decimal places: the usual local convergence of Newton--Raphson is faster.

Newton--Raphson is generally second-order convergent, whereas fixed-point iteration is generally first-order, so it should reach the accuracy faster.

(1)

(d) The next Newton--Raphson calculation requires $\ln(x_1)$.

Since $x_1 = -0.40343$, $\ln(x_1)$ is undefined over the reals, so the spreadsheet returns `#NUM!`.

(1)

(e) Iterate the supplied rearrangement, checking that the values approach one of the roots.

n	0	1	2	3	4	5	6
x_n	0.5	1.0739769	1.4524673	1.6245304	1.6932631	1.7194743	1.7293015

The iteration converges to β, rather than to α.

(2)

(f) Relaxation forms a weighted step between x_n and $g(x_n)$. Substituting the negative λ gives the required sequence.

$$0.5,\ 0.4764669,\ 0.4528879,\ 0.4293074,\ 0.4057756,\ 0.3823498,\ \ldots,$$
$$0.1116318,\ 0.1111278,\ 0.1110835,\ 0.1110821.$$

$\boxed{\alpha = 0.111082}$ *(6 d.p.).*

(2)

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Question 7

Question 7

Sarah approximates $\int_0^1 \sqrt{\tanh x}\,dx$. Her trapezium-rule results include

n	1	2	4	8	16	32
T_n	0.43634681	0.55806940	0.60199843	0.61787073	0.62357601	0.62561716

The corresponding Simpson values S_{2n}, for $n = 1, 2, 4, 8, 16$, are $0.59864360, 0.61664144, 0.62316150, 0.62547777, 0.62629755$. The midpoint value supplied for $n = 32$ is $M_{32} = 0.62707259$. Answer: **(a)** h for T_{16}; **(b)** the integral from the trapezium estimates; **(c)** convergence order; **(d)** an efficient spreadsheet formula for S_{16}; **(e)** the missing Simpson, difference and ratio entries for $n = 32$; **(f)** an extrapolated value of the integral. [1,1,2,2,3,6]

Model solution

(a) For a composite rule on $[0,1]$ using n strips, $h = (1-0)/n$.

$$h = \frac{1}{16} = 0.0625.$$

(1)

(b) The values are increasing and $T_{16} = 0.62357601$, $T_{32} = 0.62561716$ agree in their first decimal place.

0.6 is certain (and 0.63 is a plausible two-decimal estimate).

(1)

(c) On halving h, the ratio of successive changes is about 0.36. First-order convergence would give a ratio near 0.5, and second-order convergence a ratio near 0.25.

Since $0.25 < 0.36 < 0.5$, the order is between first and second order.

(2)

(d) Simpson's rule can be formed efficiently by combining the midpoint and trapezium estimates based on the same n: $S_{2n} = (2M_n + T_n)/3$.

In the displayed spreadsheet, use `=(2*O5+N5)/3`.

(2)

(e) The supplied midpoint value for $n = 32$ is combined with T_{32} to form S_{64}. The new difference is compared with the preceding Simpson difference.

$$M_{32} = 0.62707259, \qquad S_{64} = \frac{2(0.62707259) + 0.62561716}{3} = 0.62658745.$$

$$\textit{difference} = 0.62658745 - 0.62629755 = 0.00028990 \approx 0.00029,$$

$$\textit{ratio} = \frac{0.00028990}{0.62629755 - 0.62547777} = 0.3536\ldots \approx 0.354.$$

$$S_{64} = 0.62658745, \text{ difference} \approx 0.00029, \text{ ratio} \approx 0.354.$$

(3)

(f) *The Simpson sequence is converging with a nearly constant error ratio* r*, about* 0.35*--*0.36*. If the current difference is* $d = S_{64} - S_{32}$*, the remaining tail of the geometric error changes is* $d(r + r^2 + \cdots) = dr/(1 - r)$*.*

$$I \approx S_{64} + \frac{dr}{1-r} = 0.62658745 + 0.00028990\left(\frac{0.354}{1-0.354}\right) = 0.626746\ldots$$

Changing the reasonable estimate of r *between* 0.35 *and* 0.36 *leaves the first four decimal places unchanged.*

$$\boxed{\int_0^1 \sqrt{\tanh x}\,dx \approx 0.6267}, \text{ secure to 4 decimal places.}$$

(6)

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