Maths with Matt

Physics and maths resources

OCR MEI A Level Further Mathematics

H645/Y435 - Extra Pure

Thursday 21 October 2021

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the OCR MEI 2021 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

(a) A surface S is defined by $z = f(x, y)$, where

$$f(x, y) = x^3 + x^2y - 2y^2.$$

Sketch the section $z = f(2, y)$, giving the coordinates of turning points and intercepts.
(b) Find the stationary points on S.

Model solution

(a)

For this section x is fixed at 2, so the three-dimensional surface becomes a curve in the y-z plane. Substituting $x = 2$ gives a quadratic, so completing the square reveals its turning point.

$$z = f(2, y) = 2^3 + 2^2y - y^2 = 8 + 4y - 2y^2 = 10 - 2(y - 1)^2.$$

The negative coefficient of the squared term shows that this is a downward-opening parabola. Its maximum is at $y = 1$, where $z = 10$.

$$\textit{Turning point: } (y, z) = (1, 10).$$

For the z-axis intercept put $y = 0$; for the y-axis intercepts put $z = 0$.

$$z = 8 \quad \textit{when } y = 0,$$

$$8 + 4y - 2y^2 = 0 \iff y^2 - 2y - 4 = 0 \iff y = 1 \pm \sqrt{5}.$$

Thus the required sketch is a $\cap$-shaped parabola through $(0, 8)$, crossing the y-axis at $(1 - \sqrt{5}, 0)$ and $(1 + \sqrt{5}, 0)$, with maximum $(1, 10)$.

Maximum $(1, 10)$; intercepts $(0, 8)$ and $(1 \pm \sqrt{5}, 0)$.

(4)

(b)

A stationary point of a surface occurs where neither a small change in x nor a small change in y gives a first-order change in z. Therefore both first partial derivatives must be zero.

$$\frac{\partial z}{\partial x} = 3x^2 + 2xy = x(3x + 2y), \qquad \frac{\partial z}{\partial y} = x^2 - 4y.$$

$$x(3x + 2y) = 0, \qquad x^2 - 4y = 0.$$

The first equation naturally splits into two cases.

$$x = 0 \implies y = 0, \qquad z = f(0, 0) = 0.$$

$$3x + 2y = 0 \implies y = -\frac{3}{2}x.$$

Substituting this second condition into $x^2 - 4y = 0$ *determines the non-zero solution.*

$$x^2 - 4\left(-\frac{3}{2}x\right) = 0 \iff x(x+6) = 0.$$

$$x = -6, \qquad y = -\frac{3}{2}(-6) = 9,$$

$$z = (-6)^3 + (-6)^2(9) - 2(9)^2 = -216 + 324 - 162 = -54.$$

The stationary points are $(0,0,0)$ *and* $(-6,9,-54)$.

(7)

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Question 2

Question 2

Let G be a group of order 8. (a) Explain why G has no subgroup of order 6.
Now let $G = \langle g \rangle$ be cyclic, with identity e, and let H be its subgroup of order 4.
(b) Write down the possible generators of H.
The group $(\{0, 1, \ldots, 7\}, +_8)$ is isomorphic to G.
(c) Specify all possible isomorphisms between these groups.

Model solution

(a)

Lagrange's theorem says that the order of a subgroup must divide the order of its finite group.

$$6 \nmid 8.$$

There is no subgroup of order 6.

(1)

(b)

The unique subgroup of order 4 *is* $H = \langle g^2 \rangle = \{e, g^2, g^4, g^6\}$. *In a cyclic group of order* 4, *precisely the elements of order* 4 *are generators.*

g^2 *and* g^6.

(2)

(c)

An isomorphism must send the identity 0 *to* e. *It is determined by the image of* 1, *which must be a generator of* G. *The generators of* G *are* g, g^3, g^5, g^7. *If* $1 \mapsto g^k$, *then* $r \mapsto g^{kr}$, *with exponents reduced modulo* 8.

r	0	1	2	3	4	5	6	7
$\phi_1(r)$	e	g	g^2	g^3	g^4	g^5	g^6	g^7
$\phi_3(r)$	e	g^3	g^6	g	g^4	g^7	g^2	g^5
$\phi_5(r)$	e	g^5	g^2	g^7	g^4	g	g^6	g^3
$\phi_7(r)$	e	g^7	g^6	g^5	g^4	g^3	g^2	g

The four maps $\phi_k : r \mapsto g^{kr}$ *for* $k = 1, 3, 5, 7$ *are all the isomorphisms.*

(4)

Question 3

Question 3

$$A = \begin{pmatrix} 3 & 3 & 0 \\ 0 & 2 & 2 \\ 1 & 3 & 4 \end{pmatrix}.$$

(**a**) Determine the characteristic equation of A. (**b**) Hence verify that the eigenvalues are 1, 2 and 6. (**c**) For each eigenvalue determine an associated eigenvector. (**d**) Find A^n as a single matrix, where n is positive.

Model solution

(a)

Eigenvalues are found by setting $\det(A-\lambda I) = 0$. *Expanding along the first row is efficient here because of the zero entry.*

$$\begin{aligned} 0 &= \det \begin{pmatrix} 3-\lambda & 3 & 0 \\ 0 & 2-\lambda & 2 \\ 1 & 3 & 4-\lambda \end{pmatrix} \\ &= (3-\lambda)\big((2-\lambda)(4-\lambda)-6\big) - 3(0(4-\lambda)-2) \\ &= -\lambda^3 + 9\lambda^2 - 20\lambda + 12. \end{aligned}$$

$$-\lambda^3 + 9\lambda^2 - 20\lambda + 12 = 0.$$

(3)

(b)

Substitution checks each proposed eigenvalue in the characteristic equation.

$$\begin{aligned} p(1) &= -1 + 9 - 20 + 12 = 0, \\ p(2) &= -8 + 36 - 40 + 12 = 0, \\ p(6) &= -216 + 324 - 120 + 12 = 0. \end{aligned}$$

1, 2 and 6 are eigenvalues.

(1)

(c)

For each eigenvalue λ, *solve* $(A-\lambda I)\boldsymbol{v} = \boldsymbol{0}$. *Choosing a convenient free variable gives a non-zero eigenvector.*

$$\lambda = 1: \quad 2a + 3b = 0, \quad b + 2c = 0 \implies \boldsymbol{v}_1 = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}.$$

$$\lambda = 2: \quad a + 3b = 0, \quad c = 0 \implies \boldsymbol{v}_2 = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}.$$

$$\lambda = 6: \quad a = b, \quad c = 2b \implies \boldsymbol{v}_6 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

$\boldsymbol{v}_1 = (3,-2,1)^T$, $\boldsymbol{v}_2 = (-3,1,0)^T$, $\boldsymbol{v}_6 = (1,1,2)^T$ *(or non-zero multiples).*

(4)

(d)

The three distinct eigenvalues give three independent eigenvectors, so A can be diagonalised. This turns the power A^n into powers of the individual eigenvalues.

$$E = \begin{pmatrix} 3 & -3 & 1 \\ -2 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}, \qquad E^{-1} = \frac{1}{10}\begin{pmatrix} -2 & -6 & 4 \\ -5 & -5 & 5 \\ 1 & 3 & 3 \end{pmatrix},$$

$$A^n = E\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 6^n \end{pmatrix} E^{-1}.$$

Multiplying these matrices gives one matrix valid for every positive integer n.

$$A^n = \frac{1}{10}\begin{pmatrix} -6+15\,2^n+6^n & -18+15\,2^n+3\,6^n & 12-15\,2^n+3\,6^n \\ 4-5\,2^n+6^n & 12-5\,2^n+3\,6^n & -8+5\,2^n+3\,6^n \\ -2+2\,6^n & -6+6\,6^n & 4+6\,6^n \end{pmatrix}.$$

A^n *is the matrix displayed above.*

(6)

Question 4

Question 4

$$u_{n+2} - 3u_{n+1} - 10u_n = 24n - 10.$$

(a) Determine the general solution. **(b)** Find the particular solution for $u_0 = 6$, $u_1 = 10$. **(c)** Check directly that it gives the correct u_2. **(d)** If $v_n = \frac{u_n}{p^n}$ converges to a finite non-zero limit q, find p and q.

Model solution

(a)

First solve the associated homogeneous recurrence. The right-hand side is linear in n, so a linear trial function is appropriate for a particular solution.

$$r^2 - 3r - 10 = 0 \iff (r-5)(r+2) = 0,$$

$$u_n^{(h)} = \alpha 5^n + \beta(-2)^n.$$

$$u_n^{(p)} = an + b.$$

$$a(n+2) + b - 3[a(n+1) + b] - 10(an + b) = 24n - 10.$$

$$-12a = 24, \qquad -a - 12b = -10,$$

$$a = -2, \qquad b = 1.$$

$$u_n = 1 - 2n + \alpha 5^n + \beta(-2)^n.$$

(6)

(b)

Use the two initial values to determine the two constants in the general solution.

$$u_0 = 1 + \alpha + \beta = 6 \implies \alpha + \beta = 5,$$

$$u_1 = -1 + 5\alpha - 2\beta = 10 \implies 5\alpha - 2\beta = 11.$$

$$2\alpha + 2\beta = 10 \implies 7\alpha = 21, \qquad \alpha = 3, \quad \beta = 2.$$

$$u_n = 1 - 2n + 3 \cdot 5^n + 2(-2)^n.$$

(3)

(c)

A direct check should obtain u_2 independently from the recurrence and from the formula.

$$u_2 = 3u_1 + 10u_0 + 24(0) - 10 = 3(10) + 10(6) - 10 = 80,$$

$$u_2 = 1 - 2(2) + 3(5^2) + 2(-2)^2 = 1 - 4 + 75 + 8 = 80.$$

Both methods give $u_2 = 80$.

(1)

(d)

Write v_n so that the exponential bases are visible. A finite non-zero limit requires the dominant term 5^n to be exactly cancelled by p^n.

$$v_n = \left(\frac{1}{p}\right)^n (1 - 2n + 3 \cdot 5^n + 2(-2)^n) = \left(\frac{1}{p}\right)^n (1 - 2n) + 3\left(\frac{5}{p}\right)^n + 2\left(-\frac{2}{p}\right)^n.$$

Here $p \neq 0$. If $|p| > 5$, every term tends to zero. If $0 < |p| < 5$, the term $3(5/p)^n$, whose modulus grows fastest, prevents a finite limit. For $p = -5$ this term oscillates, while for $p = 5$ it is constant. Hence only $p = 5$ can give a finite non-zero limit.

$$v_n = \left(\frac{1}{5}\right)^n (1 - 2n) + 3 + 2\left(-\frac{2}{5}\right)^n \longrightarrow 3.$$

$p = 5$ and $q = 3$.

(4)

Question 5

Question 5

For $z \geqslant 0$, the surface S is $x^2 + y^2 + 2z^2 = 126$. The tangent plane at a point of S meets the x-y plane at an angle $\pi/3$. Show that the set C of such points lies in a plane P, and determine P.

Model solution

Writing the surface implicitly makes its gradient available. The gradient is normal to the tangent plane, while $(0,0,1)$ is normal to the x-y plane; hence the angle between the planes can be found from these normals.

$$g(x,y,z) = x^2 + y^2 + 2z^2, \qquad \nabla g = (2x, 2y, 4z).$$

$$\cos\frac{\pi}{3} = \frac{|\nabla g \cdot (0,0,1)|}{|\nabla g||(0,0,1)|} = \frac{4z}{\sqrt{4x^2+4y^2+16z^2}},$$

Here $z \geqslant 0$, so $|4z| = 4z$. Squaring now removes the square root.

$$\frac{1}{4} = \frac{16z^2}{4x^2+4y^2+16z^2} \iff x^2 + y^2 = 12z^2.$$

Use the equation of the surface to eliminate $x^2 + y^2$.

$$12z^2 + 2z^2 = 126 \implies z^2 = 9.$$

$$z \geqslant 0 \implies z = 3.$$

C lies in the plane P : $z = 3$.

(6)

Question 6

Question 6

Let $q \in \mathbb{Z}$ with $q \geqslant 1$, and let

$$S = \frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \frac{1}{(q+1)(q+2)(q+3)} + \cdots .$$

(a) Show that $S < 1/q$. **(b)** Deduce that $S \notin \mathbb{Z}$. **(c)** Given $e = \sum_{r=0}^{\infty} 1/r!$, prove by contradiction that e is irrational.

Model solution

(a)

Each successive denominator of S is at least as large as the corresponding power of $q+1$. This lets us compare S with a geometric series.

$$S < \frac{1}{q+1} + \frac{1}{(q+1)^2} + \frac{1}{(q+1)^3} + \cdots .$$

$$S < \frac{\frac{1}{q+1}}{1-\frac{1}{q+1}} = \frac{1}{q}.$$

$$S < 1/q.$$

(3)

(b)

All terms of S are positive, and $q \geqslant 1$ gives $1/q \leqslant 1$. Therefore S is strictly between two consecutive integers.

$$0 < S < \frac{1}{q} \leqslant 1.$$

$$S \notin \mathbb{Z}.$$

(2)

(c)

Assume, for a contradiction, that $e = p/q$ for positive integers p, q. Multiplying by $q!$ is the key step: it makes all early factorial terms integers, leaving the non-integer tail S.

$$q!e = p(q-1)! = \sum_{r=0}^{\infty} \frac{q!}{r!}.$$

$$\begin{aligned} p(q-1)! &= \sum_{r=0}^{q-1} \frac{q!}{r!} + \frac{q!}{q!} + \frac{q!}{(q+1)!} + \frac{q!}{(q+2)!} + \cdots \\ &= \sum_{r=0}^{q-1} \frac{q!}{r!} + 1 + \frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \cdots \\ &= \sum_{r=0}^{q-1} \frac{q!}{r!} + 1 + S. \end{aligned}$$

The left-hand side and the finite sum on the right are integers, so this equality would force S to be an integer. Part (b) proves that it is not, which is the contradiction.

Hence e is irrational.

(3)

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