Maths with Matt

Physics and maths resources

OCR MEI A Level Further Mathematics

H645/Y436 - Further Pure with Technology

Friday 22 October 2021

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the OCR MEI 2021 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

A family of circles is

$$(x - 2\cos a)^2 + (y - 2\sin a)^2 = 1, \qquad 0 \leqslant a < 2\pi.$$

Answer parts (a)–(e) concerning the locus of the centres, the distance between two centres, touching circles, and the envelope.

Model solution

(a) The centre is read directly from the completed-square form of the circle. Its coordinates are $(2\cos a, 2\sin a)$*, so eliminating the parameter with* $\cos^2 a + \sin^2 a = 1$ *gives*

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = 1.$$

$$x^2 + y^2 = 4.$$

(1)

(b) The two centres lie on the circle of radius 2 *at angles* b *and* c*. The included angle is* $c - b$*, so the cosine rule is the natural way to find their separation* d*.*

$$d^2 = 2^2 + 2^2 - 2(2)(2)\cos(c - b) = 8\{1 - \cos(c - b)\}.$$

$$d = 2\sqrt{2\{1 - \cos(c - b)\}}.$$

(2)

(c) Each circle has radius 1*. Two equal circles touch externally exactly when their centres are distance* 2 *apart. Substitute* $d = 2$ *in the result from part (b).*

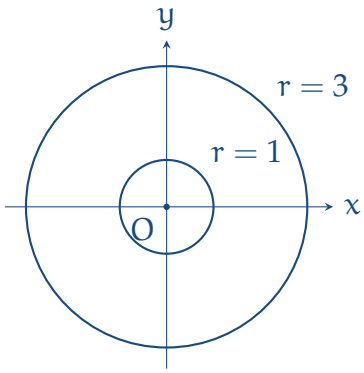
$$4 = 8\{1 - \cos(c - b)\} \quad \Rightarrow \quad \cos(c - b) = \frac{1}{2}.$$

Since $0 \leqslant b < c < \pi$*, we have* $0 < c - b < \pi$*, so the only possible angle is* $\pi/3$*.*

$$c - b = \frac{\pi}{3} \text{ (equivalently } b - c = -\pi/3\text{).}$$

(2)

(d) As the centre moves round the circle of radius 2*, every member has radius* 1*. The limiting tangent curves are therefore the concentric circles of radii* $2 - 1 = 1$ *and* $2 + 1 = 3$*. The sketch should show both complete circles, centred at the origin.*



Two concentric circles centred at the origin, with radii 1 and 3.

(2)

(e) Translating the two radii from the sketch into Cartesian equations gives the two components of the envelope.

$x^2 + y^2 = 1$ *and* $x^2 + y^2 = 9$.

(2)

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Question 2

Question 2

The line in a family passes through $(0, a)$ and $(1, a^2)$, where a is real. Find its equation, investigate parallel members, and find and sketch the envelope.

Model solution

(a) Use the two given points to find the gradient before using the point $(0, a)$.

$$m = \frac{a^2 - a}{1 - 0} = a(a-1), \qquad y - a = a(a-1)(x - 0).$$

$$y = a(a-1)x + a.$$

(2)

(b) Parallel lines have equal gradients. The parameters are distinct, so the factor $b - c$ *which occurs below cannot be zero.*

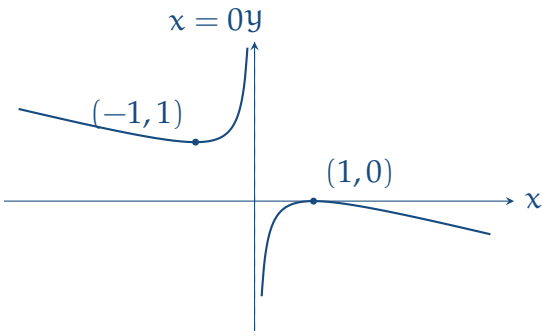
$$b(b-1) = c(c-1) \Rightarrow b^2 - b - c^2 + c = 0 \Rightarrow (b-c)(b+c-1) = 0.$$

$$b \neq c \quad \Rightarrow \quad b + c - 1 = 0.$$

$$b = 1 - c.$$

(3)

(c) The envelope has two branches. It has vertical asymptote $x = 0$, *crosses the* x*-axis at* $(1, 0)$, *and has the branch for* $x > 0$ *at or below the axis and the branch for* $x < 0$ *above the axis. These features give the required sketch.*



Sketch $y = -(x-1)^2/(4x)$ *with vertical asymptote* $x = 0$.

(2)

(d) For an envelope, neighbouring members of the family meet at a limiting point. Thus, writing the family as $f(x, y, a) = 0$, *solve both* $f = 0$ *and* $\partial f/\partial a = 0$.

$$f(x, y, a) = a(a-1)x + a - y, \qquad \frac{\partial f}{\partial a} = (2a-1)x + 1.$$

The derivative equation gives the parameter of the member tangent at (x, y).

$$(2a-1)x+1=0 \Rightarrow a=\frac{x-1}{2x} \qquad (x \neq 0).$$

Substitute this value into the line equation to remove a.

$$y=\frac{x-1}{2x}\left(\frac{x-1}{2x}-1\right)x+\frac{x-1}{2x}=-\frac{(x-1)^2}{4x}.$$

$$y=-\frac{(x-1)^2}{4x},\ x \neq 0.$$

(5)

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Question 3

Question 3

This question concerns highest common factors, Euler's totient function $\phi(n)$, and the number $F(k)$ of distinct fractions between 0 and 1 with denominator at most k. Write the requested programs and determine the stated values and formulae.

Model solution

(a)(i) A direct program can test every possible divisor up to the smaller input. Whenever a common divisor is found, replace the stored answer; the last one stored is the highest common factor.

```
def hcf(m, n):
    limit = min(m, n)
    answer = 1
    for i in range(1, limit + 1):
        if m % i == 0 and n % i == 0:
            answer = i
    return answer
```

(3)

(a)(ii) Running this function on the two specified integers gives

$$(74333, 89817) = 49.$$

(1)

(b)(i) By definition, $\phi(n)$ counts the integers from 1 to $n-1$ whose highest common factor with n is 1. Reuse the function from part (a) and increment a counter precisely in that case.

```
def phi(n):
    count = 0
    for m in range(1, n):
        if hcf(m, n) == 1:
            count = count + 1
    return count
```

(3)

(b)(ii) Evaluating the program gives

$$\phi(128) = 64 \text{ and } \phi(1000) = 400.$$

(2)

(b)(iii) A number less than 2^n is coprime to 2^n exactly when it is odd: every even number shares the factor 2, while no odd number does. There are 2^{n-1} odd numbers in $1, 2, \ldots, 2^n - 1$.

$$\phi(2^n) = 2^{n-1}.$$

(2)

(b)(iv) A number is coprime to 10^n exactly when its final digit is 1, 3, 7 or 9; the other final digits have a common factor 2 or 5 with 10^n. Each block of ten consecutive integers contributes four such numbers, and there are 10^{n-1} blocks before 10^n.

$$\phi(10^n) = 4 \times 10^{n-1}.$$

(3)

(c)(i) List each fraction only once in lowest terms. This makes it clear that equivalent fractions, such as $2/4$ and $1/2$, are not counted twice.

$$F(5) = 9: \quad \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5},$$

$$F(6) = 11: \quad \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}.$$

$$F(5) = 9,\ F(6) = 11.$$

(2)

(c)(ii) Moving from denominator at most l to denominator at most $l+1$, the new fractions have denominator $l+1$ in lowest terms. Their numerators are exactly the integers coprime to $l+1$, and there are $\phi(l+1)$ of them.

$$F(l+1) = F(l) + \phi(l+1).$$

(2)

(c)(iii) Starting with $F(1) = 1$, repeatedly apply part (ii). Equivalently, sum the totients through 100.

$$F(100) = \sum_{k=1}^{100} \phi(k).$$

A short adaptation of the program in part (b), adding `phi(k)` for $k = 1$ to 100, gives

```
def fracs(k):
    count = 0
    for i in range(1, k + 1):
        count = count + phi(i)
    return count
```

$$F(100) = 3043.$$

(2)

Question 4

Question 4

Consider

$$\frac{dy}{dx} = \frac{1-x}{2(x+1)} + a\arctan y,$$

where a is constant. Solve and sketch the $a = 0$ case, interpret the supplied tangent fields, and use modified Euler's method in a spreadsheet.

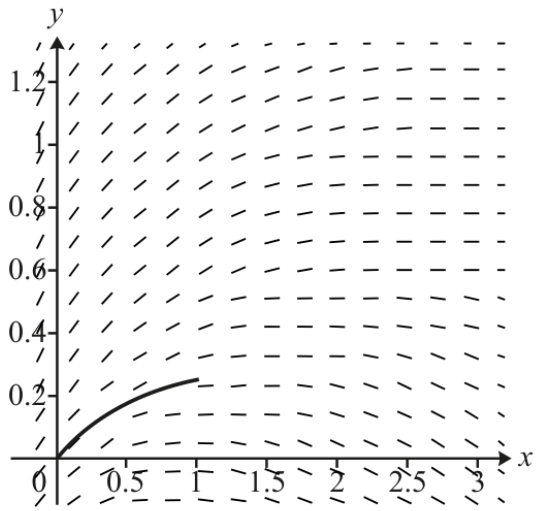


Fig 4.1

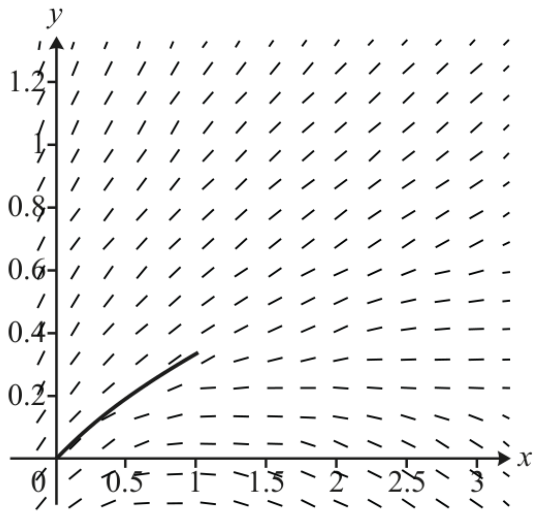


Fig 4.2

Model solution

(a)(i) When $a = 0$, the right-hand side depends only on x, so integrate directly and then use $y(0) = 0$.

$$\frac{dy}{dx} = \frac{1-x}{2(x+1)} = -\frac{1}{2} + \frac{1}{x+1},$$

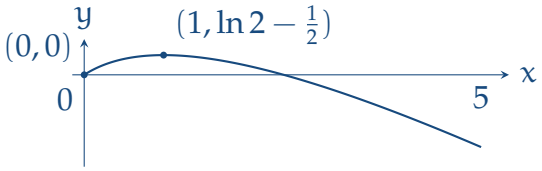
$$y = -\frac{x}{2} + \ln(x+1) + C.$$

$$0 = \ln 1 - 0 + C \Rightarrow C = 0.$$

$$y = \ln(x+1) - \frac{x}{2}.$$

(1)

(a)(ii) The curve starts at $(0,0)$, initially rises, reaches one maximum, then falls. It passes through $(3, \ln 4 - 3/2)$ and is below the axis by $x = 5$; these features fix the required sketch on $0 \leqslant x \leqslant 5$.



A curve through $(0,0)$ with a maximum at $x = 1$, then decreasing for $1 < x \leqslant 5$.

(1)

(a)(iii) Differentiate the explicit solution to locate its stationary point.

$$\frac{dy}{dx} = \frac{1}{x+1} - \frac{1}{2}.$$

$$\frac{1}{x+1} - \frac{1}{2} = 0 \Rightarrow x = 1.$$

The derivative changes from positive to negative at $x = 1$, so this stationary point is the maximum.

$$y_{max} = \ln 2 - \frac{1}{2}, \text{ at } x = 1.$$

(2)

(b)(i) In Fig. 4.1, compare the displayed slopes with the differential equation; a suitable slider value is

$$a \approx 0.5.$$

(1)

(b)(ii) The stronger positive contribution from $a \arctan y$ in Fig. 4.2 corresponds to

$$a \approx 1.$$

(1)

(b)(iii) Continue each curve by following the tangent directions. In Fig. 4.1 the solution turns down after a single local maximum; in Fig. 4.2 it continues increasing over $1 \leqslant x \leqslant 3$.

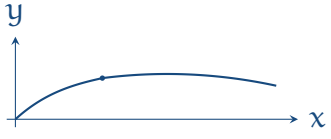


Fig. 4.1

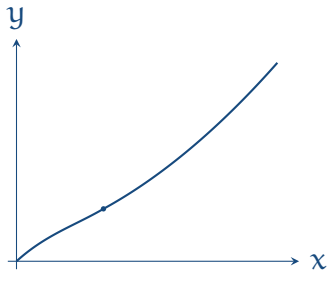


Fig. 4.2

Extend Fig. 4.1 through a maximum and downward; extend Fig. 4.2 as an increasing curve.

(2)

(b)(iv) The distinguishing feature is therefore the presence of a local maximum in only one curve.

> *Fig. 4.1 has a stationary point (a local maximum), whereas Fig. 4.2 is increasing for the displayed values of x.*

(1)

(c)(i) Use the stated modified Euler (improved Euler) method. Put the adjustable step size h in I1 and the adjustable value of a in K1; columns A–D hold x_n, y_n, k_1, k_2. The absolute references ensure that h and a remain fixed when the formulae are filled down. Start with $A1 = 0$ and $B1 = 0$.

```
C1 =$I$1*((1-A1)/(2*(A1+1))+$K$1*ATAN(B1))
D1 =$I$1*((1-(A1+$I$1))/(2*(A1+$I$1+1))
     +$K$1*ATAN(B1+C1))
A2 =A1+$I$1
B2 =B1+0.5*(C1+D1)
```

Copy the formulae in row 2 downwards (and calculate C2, D2 using the same row-relative forms) until $x = 5$. Here $C_n = k_1$ and $D_n = k_2$, so the last formula is exactly $y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$. **(4)**

(c)(ii) With $a = 0.5$ and $h = 0.1$, the spreadsheet value in the row $x = 5.0$ is

$$y(5) \approx -0.249889.$$

(1)

(c)(iii) Repeating with $a = 1$ and the same step size gives

$$y(5) \approx 3.160809.$$

(1)

(c)(iv) Search near the transition value using $h = 0.1$. The two trials below lie on opposite sides of the stated behaviour, so they bracket c.

a	*behaviour of the numerical solution from $(0, 0)$*
0.645	*increases initially, then reaches a peak and decreases*
0.655	*increases without bound*

Thus $0.645 < c < 0.655$, which rounds to 0.65 to two decimal places.

$$c = 0.65 \text{ (to 2 d.p.).}$$

(4)