Maths with Matt

Physics and maths resources

Edexcel OCR MEI A Level Further Mathematics

H645/Y420 - Core Pure

Wednesday 25 May 2022

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2022 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

(a) Find $\sum_{r=1}^{n}(3r^2+3r+1)$ by considering $(r+1)^3-r^3$. (b) Hence find $\sum_{r=1}^{n} r(r+1)$.

Model solution

(a) Expanding consecutive cubes creates a telescoping sum. $(r+1)^3-r^3=3r^2+3r+1$, *so*

$$\sum_{r=1}^{n}(3r^2+3r+1)=2^3-1^3+\cdots+(n+1)^3-n^3=(n+1)^3-1.$$

$$\sum_{r=1}^{n}(3r^2+3r+1)=(n+1)^3-1.$$

(3) (b) Isolate the required sum from the result just found.

$$3\sum_{r=1}^{n} r(r+1)+n=(n+1)^3-1, \quad \sum_{r=1}^{n} r(r+1)=\frac{(n+1)^3-1-n}{3}=\frac{n(n+1)(n+2)}{3}.$$

$$\frac{n(n+1)(n+2)}{3}.$$

(4)

Question 2

Question 2

Find exactly $\int_3^\infty \frac{1}{x^2-4x+5}\,dx$.

Model solution

As this is improper, use a limit; completing the square gives an arctangent.

$$\lim_{k\to\infty}\int_3^k \frac{dx}{(x-2)^2+1} = \lim_{k\to\infty}[\arctan(x-2)]_3^k = \frac{\pi}{2}-\frac{\pi}{4}.$$

$\frac{\pi}{4}$.

(5)

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Question 3

Question 3

Solve $3\cosh x = 2\sinh^2 x$ in exact logarithmic form.

Model solution

Use $\sinh^2 x = \cosh^2 x - 1$, ***making a quadratic in*** $\cosh x$.

$$2\cosh^2 x - 3\cosh x - 2 = 0 = (2\cosh x + 1)(\cosh x - 2).$$

$\cosh x \geqslant 1$, ***hence*** $\cosh x = 2$. ***Thus*** $e^x = 2 \pm \sqrt{3}$.

$$x = \ln(2+\sqrt{3}) \text{or} x = \ln(2-\sqrt{3}) = -\ln(2+\sqrt{3}).$$

(6)

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Question 4

Question 4

(a) The matrix $\begin{pmatrix} m & 2 & 1 \\ 0 & 1 & -2 \\ 2 & 0 & 3 \end{pmatrix}$ maps a cube into a plane. Find m. (b) $\det M = 4k+3$, $\det N = 3k+1$; NM preserves orientation and has area scale factor 2. Find k.

Model solution

(a) Images lie in a plane precisely when the transformation is singular.

$$\det \begin{pmatrix} m & 2 & 1 \\ 0 & 1 & -2 \\ 2 & 0 & 3 \end{pmatrix} = 3m - 10 = 0.$$

$$m = \tfrac{10}{3}.$$

(3) (b) Determinants multiply; preservation of orientation makes the determinant $+2$.

$$(3k+1)(4k+3) = 2 \implies 12k^2 + 13k + 1 = 0.$$

$$k = -1 \text{or} -\tfrac{1}{12}.$$

(4)

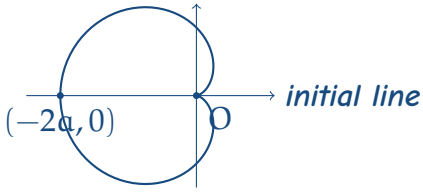
Question 5

Question 5

For $r = a(1 - \cos\theta)$, sketch the curve and find its area.

Model solution

(a) Dependence on $\cos\theta$ *gives symmetry about the initial line.* $r = 0$ *at* $\theta = 0$*, and* $r = 2a$ *at* $\theta = \pi$*.*



$$\text{A left-facing cardioid with cusp at the pole and leftmost point } (-2a, 0).$$

(2) (b) Use the polar area formula over one complete traversal.

$$A = \frac{1}{2}\int_0^{2\pi} a^2(1-\cos\theta)^2 d\theta = \frac{a^2}{4}[3\theta - 4\sin\theta + \tfrac{1}{2}\sin 2\theta]_0^{2\pi}.$$

$$A = \tfrac{3}{2}\pi a^2.$$

(5)

Question 6

Question 6

Prove by induction that $\begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}^n = \begin{pmatrix} 2^n & 0 \\ 1-2^n & 1 \end{pmatrix}$.

Model solution

For $n = 1$*, both sides are* $\begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$*. Assume the result for* $n = k$*. Then*

$$\begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 2^k & 0 \\ 1-2^k & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2^{k+1} & 0 \\ 1-2^{k+1} & 1 \end{pmatrix}.$$

This is the required form for $k+1$*.*

The identity holds for all positive integers n.

(5)

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Question 7

Question 7

Show $\int_2^3 \frac{x+1}{(x-1)(x^2+1)}dx = \frac{1}{2}\ln 2.$

Model solution

Use partial fractions, matching the linear and quadratic denominator factors.

$$\frac{x+1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}.$$

$x = 1$ *gives* $A = 1$*; comparing* x^2 *and constants gives* $B = -1, C = 0$*. Hence*

$$\int_2^3 \left(\frac{1}{x-1} - \frac{x}{x^2+1}\right) dx = [\ln(x-1) - \tfrac{1}{2}\ln(x^2+1)]_2^3 = \ln 2 - \tfrac{1}{2}\ln 2.$$

$$\tfrac{1}{2}\ln 2.$$

(9)

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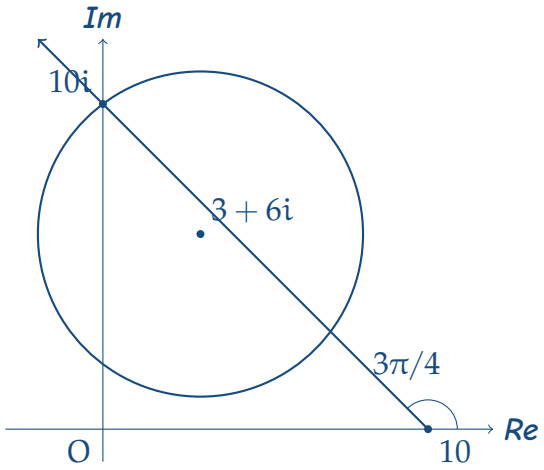
Question 8

Question 8

The loci are $\arg(z-10) = 3\pi/4$ and $|z-(3+6i)| = k$, with an intersection on the imaginary axis. Sketch and find the intersections.

Model solution

(a) The first locus is the half-line through 10 *at* $135°$*; the second is a circle centred at* $3+6i$*. The stated intersection is* $10i$*.*



$$\text{The ray } x+y=10 \text{ and the circle centred at } 3+6i \text{ meet on the imaginary axis at } 10i.$$

(4) (b) First use $10i$ *to find the circle radius, then solve the simultaneous equations.*

$$k^2 = 3^2 + (6-10)^2 = 25, \quad (x-3)^2 + (10-x-6)^2 = 25.$$

$$2x^2 - 14x = 0, \quad x = 0 \text{ or } 7.$$

$$k = 5, \text{ and the intersections are } 10i \text{ and } 7+3i.$$

(7)

Question 9

Question 9

$f(x) = \ln(1 + \sinh x)$. Answer parts (a)–(d).

Model solution

(a) A logarithm requires its argument to be positive. $\sinh k > -1$, *so*

$$k > \sinh^{-1}(-1) = \ln(\sqrt{2} - 1).$$

(2) (b) Apply chain and quotient rules, using $\cosh^2 x - \sinh^2 x = 1$.

$$f'(x) = \frac{\cosh x}{1 + \sinh x}, \qquad f''(x) = \frac{\sinh x - 1}{(1 + \sinh x)^2}.$$

$$\text{These are } f' \text{ and } f''.$$

(5) (c) Use the quadratic Taylor polynomial at zero. $f(0) = 0, f'(0) = 1, f''(0) = -1$.

$$f(x) \approx x - \tfrac{1}{2}x^2.$$

(3) (d) The approximation at 0.1 *is* 0.095.

$$\frac{\ln(1 + \sinh 0.1) - 0.095}{\ln(1 + \sinh 0.1)} 100\% = 0.48\%.$$

$$0.48\%.$$

(2)

Question 10

Question 10

The roots of $4x^4 + 16x^3 + ax^2 + bx + 6 = 0$ are $\alpha, \frac{2}{\alpha}, \beta, 3\beta$, where $\beta < 0$. Find the roots and then a, b.

Model solution

(a) Vieta's product gives $\alpha \cdot \frac{2}{\alpha} \cdot \beta \cdot 3\beta = \frac{6}{4}$, *so* $\beta = -\frac{1}{2}$. *The sum condition gives* $\alpha + \frac{2}{\alpha} + 4\beta = -4$, *hence* $\alpha^2 + 2\alpha + 2 = 0$.

$$\alpha = -1 \pm i.$$

$$\text{Roots} : -1 + i, -1 - i, -\tfrac{1}{2}, -\tfrac{3}{2}.$$

(6) (b) Multiply the linear factors and compare coefficients.

$$(x+1+i)(x+1-i)(2x+1)(2x+3) = (x^2+2x+2)(4x^2+8x+3) = 4x^4+16x^3+27x^2+22x+6.$$

$$a = 27, \ b = 22.$$

(4)

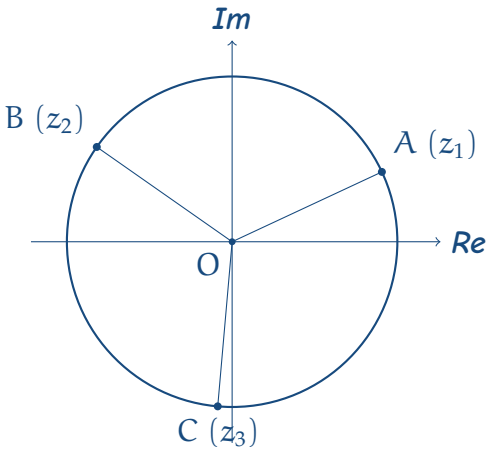
Question 11

Question 11

$z_2 = z_1 e^{2\pi i/3}, z_3 = z_1 e^{4\pi i/3}$. Answer parts (a)–(b).

Model solution

(a)(i) Multiplication by $e^{i\phi}$ rotates by ϕ.



B, C are the points obtained from A by anticlockwise rotations of $120°, 240°$, formir

(2) (a)(ii) Factor out z_1 and use the cube roots of unity.

$$z_1 + z_2 + z_3 = z_1(1 + e^{2\pi i/3} + e^{4\pi i/3}) = 0.$$

$$z_1 + z_2 + z_3 = 0.$$

(2) (b) Write $8i = 8e^{i\pi/2}$, then take cube roots.

$$z = 2e^{i(\pi/6 + 2\pi r/3)}, \quad r = 0, 1, 2.$$

$$\sqrt{3} + i, -\sqrt{3} + i, -2i.$$

(4)

Question 12

Question 12

Solve $(4-x^2)y' - xy = 1$, where $y(0) = 1$.

Model solution

Put the equation in linear form and use an integrating factor.

$$y' - \frac{x}{4-x^2}y = \frac{1}{4-x^2}, \quad \text{IF} = e^{\int -x/(4-x^2)\,dx} = \sqrt{4-x^2}.$$

$$\frac{d}{dx}(\sqrt{4-x^2}y) = \frac{1}{\sqrt{4-x^2}}, \quad \sqrt{4-x^2}y = \sin^{-1}(x/2) + C.$$

$y(0) = 1$ *gives* $C = 2$.

$$y = \frac{\sin^{-1}(x/2) + 2}{\sqrt{4-x^2}}.$$

(9)

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Question 13

Question 13

For $A = (4, 0, -1)$, $B = (10, 4, -3)$, $P_1 : x - 2y = 5$, $P_2 : 2x + 3y - z = -4$, answer parts (a)–(c)(iii).

Model solution

(a) $\overrightarrow{AB} = 2(3, 2, -1)$*; use normal* $(1, -2, 0)$ *to find the complementary angle.*

$$\cos\theta = \frac{(3, 2, -1) \cdot (1, -2, 0)}{\sqrt{14}\sqrt{5}} = -\frac{1}{\sqrt{70}}.$$

$$\text{Angle with } P_1 = 6.9^\circ.$$

(4) (b) Parametrise AB*:* $\mathbf{r} = (4, 0, -1) + \lambda(3, 2, -1)$*. In* P_1*,* $4 + 3\lambda - 4\lambda = 5$*, so* $\lambda = -1$*. The point* $(1, -2, 0)$ *gives* $2 - 6 - 0 = -4$ *in* P_2*.*

$$\text{Both meet at } (1, -2, 0).$$

(5) (c)(i) Cross the two plane normals.

$$(\mathbf{i} - 2\mathbf{j}) \times (2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) = 2\mathbf{i} + \mathbf{j} + 7\mathbf{k}.$$

(1) (c)(ii) Use $|\mathbf{n}_1 \times \mathbf{n}_2| = |\mathbf{n}_1||\mathbf{n}_2| \sin\phi$*.*

$$\sqrt{54} = \sqrt{5}\sqrt{14}\sin\phi.$$

$$\phi = 61.4^\circ.$$

(3) (c)(iii) The intersection direction is $\mathbf{d} = (2, 1, 7)$*. Use point-to-line distance from* A *to line through* $C = (1, -2, 0)$*.*

$$\frac{|(3, 2, -1) \times (2, 1, 7)|}{|(2, 1, 7)|} = \frac{\sqrt{15^2 + 23^2 + 1}}{\sqrt{54}}.$$

$$3.74 \text{ units}.$$

(4)

Question 14

Question 14

(a) Find $(3-e^{2i\theta})(3-e^{-2i\theta})$. (b) Show the stated sine series sum.

Model solution

(a) Expand and pair conjugate exponentials.

$$(3-e^{2i\theta})(3-e^{-2i\theta}) = 10-3(e^{2i\theta}+e^{-2i\theta}).$$

$$10-6\cos 2\theta.$$

(2) (b) Let $C+iS = e^{i\theta}+\frac{1}{3}e^{3i\theta}+\frac{1}{9}e^{5i\theta}+\cdots$*, a geometric series.*

$$C+iS = \frac{e^{i\theta}}{1-\frac{1}{3}e^{2i\theta}} = \frac{3e^{i\theta}(3-e^{-2i\theta})}{10-6\cos 2\theta} = \frac{9(\cos\theta+i\sin\theta)-3(\cos\theta-i\sin\theta)}{10-6\cos 2\theta}.$$

Equate imaginary parts.

$$S = \frac{6\sin\theta}{5-3\cos 2\theta}.$$

(6)

Question 15

Question 15

An m-kg particle has $x(0) = 2$, $\frac{dx}{dt}(0) = 1$, restoring force $2mx$ to equilibrium. Answer parts (a)–(c)(ii).

Model solution

(a)(i) With positive away from O, the restoring force is $-2mx$.

$$m\frac{d^2x}{dt^2} = -2mx \implies \frac{d^2x}{dt^2} + 2x = 0.$$

$$\frac{d^2x}{dt^2} + 2x = 0.$$

(1) (a)(ii)--(iii) This is the SHM form with $\omega = \sqrt{2}$.

$$\text{SHM}; T = 2\pi/\sqrt{2} = \sqrt{2}\pi s.$$

(2) (a)(iv) Solve the SHM equation and use both initial conditions.

$$x = A\cos\sqrt{2}t + B\sin\sqrt{2}t, \quad A = 2, \quad \frac{dx}{dt}(0) = \sqrt{2}B = 1.$$

$$x = 2\cos\sqrt{2}t + \tfrac{\sqrt{2}}{2}\sin\sqrt{2}t.$$

(4) (a)(v) Amplitude is $\sqrt{A^2 + B^2}$.

$$\tfrac{3\sqrt{2}}{2}m.$$

(2) (b)(i) Damping opposes velocity, giving force $-2m\frac{dx}{dt}$.

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = 0.$$

(1) (b)(ii)--(iii) The auxiliary equation $\lambda^2 + 2\lambda + 2 = 0$ has $\lambda = -1 \pm i$, so the motion is oscillatory and decaying.

$$\text{Underdamped}; x = e^{-t}(A\cos t + B\sin t).$$

(4) (c)(i) Try a particular integral at the forcing frequency.

$$x_p = C\cos 2t + D\sin 2t, \quad -2C + 4D = 2, \quad -4C - 2D = 0,$$

$$C = -0.2, \ D = 0.4.$$

$$x = e^{-t}(A\cos t + B\sin t) - 0.2\cos 2t + 0.4\sin 2t.$$

$x(0) = 2$ *gives* $A = 2.2$*; differentiating and using* $\frac{dx}{dt}(0) = 1$ *gives* $B = 2.4$.

$$x = e^{-t}(2.2\cos t + 2.4\sin t) - 0.2\cos 2t + 0.4\sin 2t.$$

(7) (c)(ii) The exponentially decaying term tends to zero, leaving a sinusoid of angular frequency 2.

$$T = 2\pi/2 = \pi \approx 3.14s, consistentwiththeobservation.$$

(2)

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