Maths with Matt

Physics and maths resources

Edexcel OCR MEI A Level Further Mathematics

H645/Y435 - Extra Pure

Friday 23 June 2023

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2023 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Contents

Question 1 2

Question 2 3

Question 3 5

Question 4 6

Question 5 8

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Question 1

Question 1

A surface is defined by $z = 3x^3 + 6xy + y^2$. Determine the coordinates of any stationary points on the surface.

Model solution

At a stationary point of a surface $z = f(x, y)$, both first partial derivatives are zero. This gives two simultaneous equations for x and y.

$$\frac{\partial z}{\partial x} = 9x^2 + 6y, \qquad \frac{\partial z}{\partial y} = 6x + 2y.$$

$$9x^2 + 6y = 0, \qquad 6x + 2y = 0.$$

The second equation is linear, so use it first to eliminate y:

$$y = -3x.$$

$$9x^2 + 6(-3x) = 0 \implies 9x(x - 2) = 0.$$

$$x = 0 \quad \textit{or} \quad x = 2.$$

Substituting into $y = -3x$ gives $(x, y) = (0, 0)$ and $(2, -6)$. The corresponding z-coordinates must also be found.

$$z(0, 0) = 0,$$

$$z(2, -6) = 3(2)^3 + 6(2)(-6) + (-6)^2 = 24 - 72 + 36 = -12.$$

$$\text{The stationary points are } (0, 0, 0) \text{ and } (2, -6, -12).$$

(7)

Question 2

Question 2

(a) A sequence satisfies $4t_{n+1} - t_n = 15n + 17$ for $n \geqslant 1$, with $t_1 = 2$. Solve the recurrence relation.

(b)(i) Another sequence satisfies $(n+1)u_{n+1} - u_n^2 = 2n - \frac{1}{n^2}$, with $u_1 = 2$. Explain why standard techniques for non-homogeneous first-order recurrence relations cannot be used.

(b)(ii) Verify that $u_n = an + \frac{b}{n}$, where a, b are constants to be determined.

(c) A third sequence is defined by $v_n = \frac{t_n}{u_n}$. Determine $\lim_{n\to\infty} v_n$.

Model solution

(a) First solve the associated homogeneous recurrence; the remaining linear right-hand side suggests trying a linear particular solution.

$$4t_{n+1} - t_n = 0 \quad \Longrightarrow \quad t_{n+1} = \frac{1}{4}t_n,$$

$$t_n^{(h)} = A\left(\frac{1}{4}\right)^n.$$

Try $t_n^{(p)} = pn + q$. *Substitution into the original recurrence gives*

$$4\big(p(n+1) + q\big) - (pn + q) = 15n + 17,$$

$$3pn + 4p + 3q = 15n + 17.$$

Equating the coefficients of n *and the constants gives*

$$3p = 15, \qquad 4p + 3q = 17,$$

$$p = 5, \qquad q = -1.$$

$$t_n = 5n - 1 + A\left(\frac{1}{4}\right)^n.$$

Now use the initial value to determine the one remaining constant.

$$2 = t_1 = 5 - 1 + \frac{A}{4} \quad \Longrightarrow \quad A = -8.$$

$$t_n = 5n - 1 - 8\left(\frac{1}{4}\right)^n.$$

(7)

(b)(i) Standard first-order linear recurrence methods require the terms to be linear in the sequence values, with coefficients depending at most on n. *Here the term* u_n^2 *makes the recurrence non-linear.*

It cannot be solved by those standard techniques because it is non-linear in u_n.

(1)

(b)(ii) The proposed form contains both an n term and a reciprocal term, matching the forms on the right-hand side. Substitute it and compare coefficients.

$$u_{n+1} = a(n+1) + \frac{b}{n+1}.$$

$$\begin{aligned}(n+1)u_{n+1} - u_n^2 &= (n+1)\left(a(n+1) + \frac{b}{n+1}\right) - \left(an + \frac{b}{n}\right)^2 \\ &= a(n+1)^2 + b - a^2n^2 - 2ab - \frac{b^2}{n^2} \\ &= (a - a^2)n^2 + 2an + (a + b - 2ab) - \frac{b^2}{n^2}.\end{aligned}$$

For this to equal $2n - \frac{1}{n^2}$ for every n, corresponding coefficients must agree:

$$a - a^2 = 0, \qquad 2a = 2, \qquad a + b - 2ab = 0, \qquad b^2 = 1.$$

$$a = 1, \qquad 1 + b - 2b = 0 \Longrightarrow b = 1.$$

The initial condition is also satisfied, since $u_1 = 1 + 1 = 2$.

$$u_n = n + \frac{1}{n}.$$

(5)

(c) Use the explicit formulae. Dividing numerator and denominator by n exposes their leading terms, which determine the limit.

$$v_n = \frac{5n - 1 - 8(1/4)^n}{n + 1/n} = \frac{5 - 1/n - 8(1/4)^n/n}{1 + 1/n^2}.$$

$$\lim_{n\to\infty} v_n = \frac{5}{1} = 5.$$

$$\lim_{n\to\infty} v_n = 5.$$

(2)

Question 3

Question 3

The surface S is defined by $g(x,y,z)=0$, where $g(x,y,z)=2x^3-x^2y+2xy^2+27z$. The normal to S at $\left(1,1,-\frac{1}{9}\right)$ and the tangent plane to S at $(3,3,-3)$ intersect at P. Determine the position vector of P.

Model solution

For an implicitly defined surface, ∇g is normal to the surface. Use it first to form the normal line, and then to form the tangent plane.

$$\nabla g=\left(6x^2-2xy+2y^2,\ -x^2+4xy,\ 27\right).$$

At $A=\left(1,1,-\frac{1}{9}\right)$,

$$\nabla g(A)=(6,3,27)=3(2,1,9).$$

Thus the normal line through A *can be written as*

$$\mathbf{r}=\begin{pmatrix}1\\1\\-1/9\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\9\end{pmatrix}.$$

At $B=(3,3,-3)$, *a normal to the tangent plane is*

$$\nabla g(B)=(54,27,27)=27(2,1,1).$$

Therefore the tangent plane is

$$2(x-3)+(y-3)+(z+3)=0,$$

$$2x+y+z=6.$$

The point of intersection lies on the normal line, so substitute $x=1+2\lambda$, $y=1+\lambda$, $z=-\frac{1}{9}+9\lambda$ *into the plane equation.*

$$2(1+2\lambda)+(1+\lambda)+\left(-\frac{1}{9}+9\lambda\right)=6,$$

$$\frac{26}{9}+14\lambda=6 \quad\Longrightarrow\quad \lambda=\frac{2}{9}.$$

$$\mathbf{r}_P=\begin{pmatrix}1\\1\\-1/9\end{pmatrix}+\frac{2}{9}\begin{pmatrix}2\\1\\9\end{pmatrix}=\begin{pmatrix}13/9\\11/9\\17/9\end{pmatrix}.$$

$$\overrightarrow{OP}=\frac{1}{9}\begin{pmatrix}13\\11\\17\end{pmatrix}.$$

(8)

Question 4

Question 4

Let $G = \{M : M \text{ is a real } 2 \times 2 \text{ matrix and } \det M = 1\}$.
(a) Show that G forms a group under matrix multiplication.
(b) For an integer n, $A_n = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$, and $S = \{A_n : n \in \mathbb{Z},\ n \geqslant 0\}$. **(i)** Is S closed under multiplication? **(ii)** Is S a subgroup of $(G, \times)$?
(c) **(i)** Find a subgroup of $(G, \times)$ of order 2. **(ii)** Show that this is the only subgroup of order 2.
(d) Let H be the set of all real 2×2 matrices. With an example, explain why $(H, \times)$ is not a group.

Model solution

(a) Check the group axioms. Associativity is given, so it remains to establish closure, an identity and inverses within the set.

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in G \quad \textit{because} \quad \det I = 1,$$

$$IA = AI = A \qquad (A \in G).$$

If $A, B \in G$, *then* AB *is a real* 2×2 *matrix and*

$$\det(AB) = \det A \det B = 1 \cdot 1 = 1.$$

Thus $AB \in G$, *proving closure. Also, for* $A \in G$, $\det A = 1 \neq 0$, *so* A^{-1} *exists.*

$$1 = \det I = \det(AA^{-1}) = \det A \det(A^{-1}) \quad \Longrightarrow \quad \det(A^{-1}) = 1.$$

Hence $A^{-1} \in G$. *Together with the given associativity, all group axioms hold.*

$$(G, \times) \text{is a group.}$$

(5)

(b)(i) Multiply two general members of S; this tests closure rather than just testing a particular pair.

$$A_m A_n = \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ m+n & 1 \end{pmatrix} = A_{m+n}.$$

Since $m, n \geqslant 0$ *implies* $m + n \geqslant 0$, *this product is again in S.*

$$S \text{ is closed under multiplication.}$$

(2)

(b)(ii) A subgroup must contain the inverse of every one of its elements.

$$A_n^{-1} = \begin{pmatrix} 1 & 0 \\ -n & 1 \end{pmatrix} = A_{-n}.$$

For example, $A_1 \in S$, *but* $A_1^{-1} = A_{-1} \notin S$ *because* $-1 < 0$.

$$S \text{ is not a subgroup of } (G, \times).$$

(2)

(c)(i) The matrices I *and* $-I$ *both have determinant* 1*, and* $(-I)^2 = I$*, so they form a two-element subgroup.*

$$\{I, -I\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

One subgroup of order 2 is $\{I, -I\}$.

(2)

(c)(ii) Let M *be the non-identity member of any subgroup of order* 2*. Its inverse must be the other member of the subgroup, so* $M = M^{-1}$*. Write*

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad M^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

where this inverse formula uses $\det M = 1$*. Equality of these matrices gives*

$$a = d, \qquad b = -b, \qquad c = -c,$$

$$b = c = 0, \\qquad M = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

$$1 = \det M = a^2,$$

so $a = \pm 1$*. The value* $a = 1$ *would make* $M = I$*, contrary to choosing the non-identity member. Therefore* $M = -I$*.*

Every order $-$ 2 subgroup is $\{I, -I\}$, so this is the only one.

(2)

(d) A group requires an inverse for every element. The set H *contains singular matrices, which have no inverse. For example,*

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in H, \qquad \det \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

This matrix has no inverse, so the inverse axiom fails and $(H, \times)$ is not a group.

(2)

Question 5

Question 5

Let $P = \begin{pmatrix} a & 0 \\ 2 & 3 \end{pmatrix}$, where $a \neq 3$.

(a) The acute angle between the directions of the eigenvectors of P is $\frac{\pi}{4}$. Determine the possible values of a.

(b) Instead, suppose $I = P^2 + rP$, where r is rational. **(i)** Use the Cayley–Hamilton theorem to determine a and the corresponding r. **(ii)** Hence show that $P^4 = sI + tP$, where s, t are rational; do not calculate P^4.

Model solution

(a) As P is triangular, its eigenvalues are its diagonal entries, a and 3. They are distinct because $a \neq 3$, so find one eigenvector for each and use the dot-product angle formula.

$$P\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

so take $\mathbf{e}_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

For eigenvalue a, solve $(P - aI)\mathbf{e}_a = \mathbf{0}$:

$$\begin{pmatrix} 0 & 0 \\ 2 & 3-a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$2x + (3-a)y = 0,$$

so take $\mathbf{e}_a = \begin{pmatrix} a-3 \\ 2 \end{pmatrix}$.

The acute angle is $\pi/4$, so its cosine is $1/\sqrt{2}$.

$$\frac{|\mathbf{e}_a \cdot \mathbf{e}_3|}{|\mathbf{e}_a|\,|\mathbf{e}_3|} = \frac{2}{\sqrt{(a-3)^2+4}} = \frac{1}{\sqrt{2}}.$$

$$8 = (a-3)^2 + 4 \quad \Longrightarrow \quad (a-3)^2 = 4.$$

$$a = 1 \text{ or } a = 5.$$

(8)

(b)(i) Use Cayley--Hamilton to express P^2 as a combination of P and I, then compare it with the given relation. The characteristic polynomial is

$$(\lambda - a)(\lambda - 3) = \lambda^2 - (a+3)\lambda + 3a.$$

$$P^2 - (a+3)P + 3aI = 0 \quad \Longrightarrow \quad P^2 = (a+3)P - 3aI.$$

Since $I = P^2 + rP$, *substitution of this expression for* P^2 *gives*

$$I = (a + 3 + r)P - 3aI.$$

Matching the coefficients of P *and* I,

$$a + 3 + r = 0, \qquad -3a = 1.$$

$$a = -\frac{1}{3}, \qquad r = -(a + 3) = -\frac{8}{3}.$$

$$a = -\frac{1}{3} \text{ and } r = -\frac{8}{3}.$$

(4)

(b)(ii) The given relation now gives $P^2 = I - rP = I + \frac{8}{3}P$. *Squaring this relation and substituting for* P^2 *again avoids calculating* P^4 *from the entries of* P.

$$\begin{aligned} P^4 &= \left(I + \frac{8}{3}P\right)^2 \\ &= I + \frac{16}{3}P + \frac{64}{9}P^2 \\ &= I + \frac{16}{3}P + \frac{64}{9}\left(I + \frac{8}{3}P\right) \\ &= \frac{73}{9}I + \frac{656}{27}P. \end{aligned}$$

$$P^4 = \frac{73}{9}I + \frac{656}{27}P, \text{ so } s = \frac{73}{9} \text{ and } t = \frac{656}{27}.$$

(3)