Maths with Matt

Physics and maths resources

Edexcel OCR MEI A Level Further Mathematics

H645/Y435 - Extra Pure

Friday 21 June 2024

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2024 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

A surface S is given by $z = f(x, y) = 12x - 30y + 6xy$. (a) Determine all stationary points. (b) The section $z = f(x, a)$ is $z = 24x + b$; find a and b. (c) The contour $z = 12$ is shown with its asymptotes; find their equations. (d) With $g(x, y, z) = f(x, y) - z$, find in vector form the tangent plane at $x = 3$, $y = 2$. (e) $A = (0, 4, -120)$ lies on S. Show that its normal meets $\mathbf{r} \cdot (3, 3, -2) = 52$ at $(-360, 304, -110)$.

Model solution

(a) A stationary point on a surface has zero rate of change in both independent horizontal directions. We therefore set both first partial derivatives equal to zero.

$$\frac{\partial z}{\partial x} = 12 + 6y, \qquad \frac{\partial z}{\partial y} = -30 + 6x.$$

$$12 + 6y = 0 \Rightarrow y = -2, \qquad -30 + 6x = 0 \Rightarrow x = 5.$$

$$z = f(5, -2) = 12(5) - 30(-2) + 6(5)(-2) = 60.$$

$$\text{The only stationary point is } (5, -2, 60).$$

(5)

(b) A section with $y = a$ is found by substituting the constant value of y into the surface equation. Its given form lets us compare the coefficient of x and the constant term.

$$z = f(x, a) = 12x - 30a + 6ax = (12 + 6a)x - 30a.$$

$$(12 + 6a)x - 30a \equiv 24x + b.$$

$$12 + 6a = 24 \Rightarrow a = 2, \qquad b = -30(2) = -60.$$

$$a = 2 \text{ and } b = -60.$$

(3)

(c) The contour has $z = 12$. Rearranging its equation into y as a function of x reveals the vertical and horizontal asymptotes directly.

$$12 = 12x - 30y + 6xy \Rightarrow 2 = 2x - 5y + xy \Rightarrow y(x - 5) = 2 - 2x.$$

$$y = \frac{2 - 2x}{x - 5} = -2 - \frac{8}{x - 5}.$$

The denominator is zero at the vertical asymptote, while the remaining constant gives the limiting value as x becomes large.

The asymptotes are $x = 5$ and $y = -2$.

(3)

(d) For the implicit surface $g(x, y, z) = f(x, y) - z = 0$, ∇g is normal to the surface. This gives the normal vector needed for a vector equation of the tangent plane.

$$g = 12x - 30y + 6xy - z, \qquad \nabla g = \begin{pmatrix} 12 + 6y \\ -30 + 6x \\ -1 \end{pmatrix}.$$

$$f(3, 2) = 36 - 60 + 36 = 12, \qquad \nabla g\big|_{(3,2,12)} = \begin{pmatrix} 24 \\ -12 \\ -1 \end{pmatrix}.$$

$$\begin{pmatrix} 24 & -12 & -1 \end{pmatrix} \mathbf{r} = \begin{pmatrix} 24 & -12 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 12 \end{pmatrix} = 36.$$

The tangent plane is $\begin{pmatrix} 24 & -12 & -1 \end{pmatrix} \mathbf{r} = 36$.

(3)

(e) The normal line at $A = (0, 4, -120)$ has direction ∇g at A. We substitute its vector equation into the given plane to find its intersection parameter.

$$\nabla g\big|_{(0,4,-120)} = \begin{pmatrix} 36 \\ -30 \\ -1 \end{pmatrix}, \qquad \mathbf{r} = \begin{pmatrix} 0 \\ 4 \\ -120 \end{pmatrix} + \lambda \begin{pmatrix} 36 \\ -30 \\ -1 \end{pmatrix}.$$

$$3(36\lambda) + 3(4 - 30\lambda) - 2(-120 - \lambda) = 52.$$

$$20\lambda + 252 = 52 \Rightarrow \lambda = -10.$$

$$\mathbf{r} = \begin{pmatrix} 0 \\ 4 \\ -120 \end{pmatrix} - 10 \begin{pmatrix} 36 \\ -30 \\ -1 \end{pmatrix} = \begin{pmatrix} -360 \\ 304 \\ -110 \end{pmatrix}.$$

The normal intersects the plane at $\boxed{(-360, 304, -110)}$, as required.

(3)

Question 2

Question 2

(a) Determine the general solution of $2u_{n+2} - 7u_{n+1} + 3u_n = 0$. (b) Hence determine the general solution of $2u_{n+2} - 7u_{n+1} + 3u_n = 20n^2 + 60n$. For the sequence satisfying this recurrence, $u_0 = -9$, $u_1 = -12$. (c) Determine its particular solution. (d) Given that, once its terms start increasing, they remain increasing, determine by direct calculation the least term value.

Model solution

(a) This is a homogeneous linear recurrence, so trying $u_n = r^n$ converts it into an auxiliary quadratic.

$$2r^{n+2} - 7r^{n+1} + 3r^n = 0 \Rightarrow 2r^2 - 7r + 3 = 0 \Rightarrow (2r-1)(r-3) = 0.$$

$$r = \frac{1}{2} \quad \textit{or} \quad r = 3.$$

$$u_n = A3^n + B\left(\frac{1}{2}\right)^n.$$

(2)

(b) The non-zero right-hand side is a quadratic in n, so a quadratic trial particular solution is appropriate. It does not duplicate either complementary-solution term.

$$u_n^{(p)} = an^2 + bn + c.$$

$$\begin{aligned} &2[a(n+2)^2 + b(n+2) + c] - 7[a(n+1)^2 + b(n+1) + c] + 3(an^2 + bn + c) \\ &= -2an^2 + (-6a - 2b)n + (a - 3b - 2c). \end{aligned}$$

Equating this to $20n^2 + 60n$ makes corresponding coefficients equal.

$$-2a = 20, \qquad -6a - 2b = 60, \qquad a - 3b - 2c = 0.$$

$$a = -10, \qquad b = 0, \qquad c = -5.$$

$$u_n = A3^n + B\left(\frac{1}{2}\right)^n - 10n^2 - 5.$$

(5)

(c) Substitute the two initial values into the general solution; this determines the two arbitrary constants.

$$n = 0: \quad -9 = A + B - 5 \Rightarrow A + B = -4.$$

$$n = 1: \quad -12 = 3A + \frac{1}{2}B - 15 \Rightarrow 6A + B = 6.$$

$$5A = 10 \Rightarrow A = 2, \qquad B = -6.$$

$$u_n = 2 \cdot 3^n - 6\left(\frac{1}{2}\right)^n - 10n^2 - 5.$$

(3)

(d) We are told that once the terms begin increasing they continue to do so. Thus it is enough to calculate consecutive terms around the change from decreasing to increasing.

$$u_2 = 2(9) - 6\left(\frac{1}{4}\right) - 10(4) - 5 = -28.5,$$

$$u_3 = 2(27) - 6\left(\frac{1}{8}\right) - 10(9) - 5 = -41.75,$$

$$u_4 = 2(81) - 6\left(\frac{1}{16}\right) - 10(16) - 5 = -3.375.$$

Since $u_2 > u_3 < u_4$, the terms have started increasing after u_3; the stated condition then rules out a later smaller value.

$$\text{The least value is } u_3 = -41.75.$$

(2)

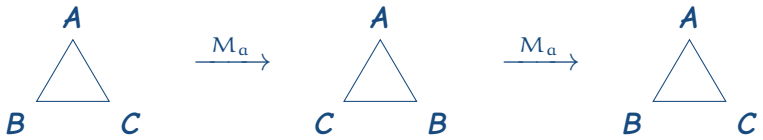
Question 3

Question 3

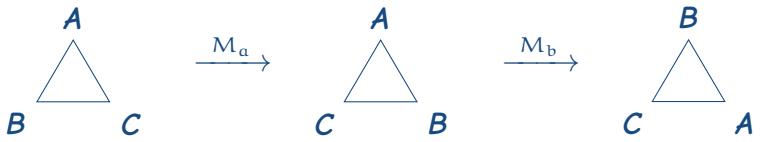
An equilateral triangle has reflection axes S_a, S_b, S_c, with reflections M_a, M_b, M_c, identity I, and anticlockwise rotations R_{120}, R_{240}. Composition $X \circ Y$ means Y followed by X. (a) Find $M_a \circ M_a$, $M_b \circ M_a$, and $R_{120} \circ M_a$. (b) Complete the composition table for $G = \{I, M_a, M_b, M_c, R_{120}, R_{240}\}$. (c) Explain why there is no subgroup of order 4. (d) Assess the claim that a group whose proper non-trivial subgroups are all abelian is abelian. (e) Explain from element orders why G is not isomorphic to C_6.

Model solution

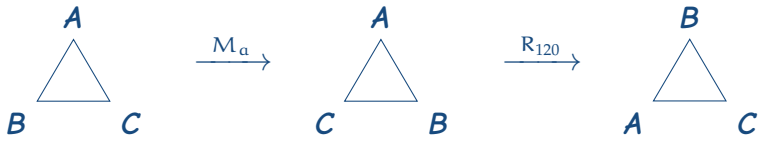
(a) In a composition $X \circ Y$, the right-hand transformation Y is applied first. The labelled diagrams therefore apply M_a first in each case, then apply the transformation on its left.



$M_a \circ M_a = I.$



$M_b \circ M_a = R_{240}.$



$R_{120} \circ M_a = M_c.$

$$M_a \circ M_a = I, \qquad M_b \circ M_a = R_{240}, \qquad R_{120} \circ M_a = M_c.$$

I, R_{240}, M_c, respectively.

(3)

(b) Each entry is the row transformation after the column transformation. The identity leaves every transformation unchanged; the remaining entries follow by composing the rotations and reflections.

$\circ$	I	M_a	M_b	M_c	R_{120}	R_{240}
I	I	M_a	M_b	M_c	R_{120}	R_{240}
M_a	M_a	I	R_{120}	R_{240}	M_b	M_c
M_b	M_b	R_{240}	I	R_{120}	M_c	M_a
M_c	M_c	R_{120}	R_{240}	I	M_a	M_b
R_{120}	R_{120}	M_c	M_a	M_b	R_{240}	I
R_{240}	R_{240}	M_b	M_c	M_a	I	R_{120}

The completed composition table is shown above.

(4)

(c) Lagrange's theorem is the relevant result because it restricts the order of every subgroup to a divisor of the order of the whole group.

$$|G| = 6, \qquad 4 \nmid 6.$$

By Lagrange's theorem no subgroup of G can have order 4.

(1)

(d) To disprove the claim, this group provides a counterexample. First identify its proper non-trivial subgroups, then compare their commutativity with that of the whole group.

$$\{I, M_a\}, \quad \{I, M_b\}, \quad \{I, M_c\}, \quad \{I, R_{120}, R_{240}\}$$

are all the proper non-trivial subgroups: by Lagrange their orders can only be 2 or 3. Groups of order 2 or 3 are abelian, so every one of these subgroups is abelian. However, the composition table is not symmetric; for example

$$M_b \circ M_a = R_{240} \neq R_{120} = M_a \circ M_b.$$

Thus G is non-abelian although all its proper non-trivial subgroups are abelian, so the claim is incorrect.

(3)

(e) Isomorphisms preserve the order of each element. Compare an order that occurs in C_6 with those available among the triangle symmetries.

C_6 has an element of order 6, whereas G has no element of order 6; hence the groups a

(1)

Question 4

Question 4

Let $P = \begin{pmatrix} 1 & 7 & 8 \\ -6 & 12 & 12 \\ -2 & 4 & 8 \end{pmatrix}$. (a) Show that its characteristic equation is $-\lambda^3 + 21\lambda^2 - 126\lambda + 216 = 0$, whose roots are $3, 6, 12$. (b)(i) Verify that $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ is an eigenvector and state its eigenvalue. (ii) An eigenvector with eigenvalue 6 is $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, where $z = 5$; find x, y. Given $P = EDE^{-1}$, where $E = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 2 & -2 \\ 1 & 1 & 2 \end{pmatrix}$ and E has characteristic equation $-\lambda^3 + 7\lambda^2 - 15\lambda + 9 = 0$: (c)(i) express E^{-1} using positive powers of E, (ii) find E^{-1}, and (iii) find P^4.

Model solution

(a) Eigenvalues come from $\det(P - \lambda I) = 0$, so form and expand this determinant.

$$\det(P - \lambda I) = \begin{vmatrix} 1-\lambda & 7 & 8 \\ -6 & 12-\lambda & 12 \\ -2 & 4 & 8-\lambda \end{vmatrix}.$$

$$\begin{aligned} &= (1-\lambda)[(12-\lambda)(8-\lambda) - 48] - 7[-6(8-\lambda) + 24] \\ &\qquad + 8[-24 + 2(12-\lambda)] \\ &= -\lambda^3 + 21\lambda^2 - 126\lambda + 216. \end{aligned}$$

$$\det(P - \lambda I) = 0 \Rightarrow -\lambda^3 + 21\lambda^2 - 126\lambda + 216 = 0.$$

The characteristic equation is $-\lambda^3 + 21\lambda^2 - 126\lambda + 216 = 0$.

(3)

(b)(i) To verify an eigenvector, multiply it by P and check whether the result is a scalar multiple of the original vector.

$$P\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ 6 \end{pmatrix} = 3\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}.$$

The vector is an eigenvector with eigenvalue 3.

(2)

(b)(ii) Use the defining equation $\mathbf{Pv} = 6\mathbf{v}$, with $z = 5$, to obtain simultaneous equations for x and y.

$$\begin{pmatrix} 1 & 7 & 8 \\ -6 & 12 & 12 \\ -2 & 4 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \\ 5 \end{pmatrix} = 6 \begin{pmatrix} x \\ y \\ 5 \end{pmatrix}.$$

$$-5x + 7y = -40, \qquad -x + y = -10.$$

$$y = x - 10 \Rightarrow -5x + 7(x - 10) = -40 \Rightarrow x = 15, \qquad y = 5.$$

$$x = 15, y = 5, \text{so one such eigenvector is } \begin{pmatrix} 15 \\ 5 \\ 5 \end{pmatrix}.$$

(3)

(c)(i) Cayley--Hamilton says that a matrix satisfies its own characteristic equation. Multiplying the resulting equation by E^{-1} isolates the inverse using only non-negative powers of E.

$$-E^3 + 7E^2 - 15E + 9I = 0.$$

$$E^2 - 7E + 15I - 9E^{-1} = 0 \Rightarrow E^{-1} = \frac{1}{9}(E^2 - 7E + 15I).$$

$$E^{-1} = \frac{1}{9}(E^2 - 7E + 15I).$$

(2)

(c)(ii) Substitute the given E into the expression from part (i).

$$E^{-1} = \frac{1}{9}\left(\begin{pmatrix} 3 & 2 & 1 \\ 1 & 2 & -2 \\ 1 & 1 & 2 \end{pmatrix}^2 - 7\begin{pmatrix} 3 & 2 & 1 \\ 1 & 2 & -2 \\ 1 & 1 & 2 \end{pmatrix} + 15I\right) = \frac{1}{9}\begin{pmatrix} 6 & -3 & -6 \\ -4 & 5 & 7 \\ -1 & -1 & 4 \end{pmatrix}.$$

$$E^{-1} = \frac{1}{9}\begin{pmatrix} 6 & -3 & -6 \\ -4 & 5 & 7 \\ -1 & -1 & 4 \end{pmatrix}.$$

(1)

(c)(iii) The columns of E are eigenvectors. Their associated eigenvalues, in column order, are $6, 12, 3$, so these form the diagonal entries of D. Then use $P^4 = ED^4E^{-1}$.

$$D = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \qquad D^4 = \begin{pmatrix} 1296 & 0 & 0 \\ 0 & 20736 & 0 \\ 0 & 0 & 81 \end{pmatrix}.$$

$$\begin{aligned}
\mathbf{P}^4 &= \mathbf{E}\mathbf{D}^4\mathbf{E}^{-1} \\
&= \begin{pmatrix} 3 & 2 & 1 \\ 1 & 2 & -2 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1296 & 0 & 0 \\ 0 & 20736 & 0 \\ 0 & 0 & 81 \end{pmatrix} \frac{1}{9} \begin{pmatrix} 6 & -3 & -6 \\ -4 & 5 & 7 \\ -1 & -1 & 4 \end{pmatrix} \\
&= \begin{pmatrix} -15849 & 21735 & 29700 \\ -17550 & 22626 & 31320 \\ -8370 & 11070 & 15336 \end{pmatrix}.
\end{aligned}$$

$$\mathbf{P}^4 = \begin{pmatrix} -15849 & 21735 & 29700 \\ -17550 & 22626 & 31320 \\ -8370 & 11070 & 15336 \end{pmatrix}.$$

(4)

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Question 5

Question 5

You may assume that for distinct primes p, q, if $p^\alpha = q^\beta$ for $\alpha, \beta \in \mathbb{Z}$, then $\alpha = \beta = 0$. (a) Prove that there are no integers a, b, with $b \neq 0$, for which $3 = 2^{a/b}$. (b) Deduce that $\log_2 3 \notin \mathbb{Q}$.

Model solution

(a) Use proof by contradiction. Raising the proposed equation to the power b gives equal integer powers of the distinct primes 2 and 3, so the stated result applies.

$$\textit{Assume } 3 = 2^{a/b} \textit{ for some } a, b \in \mathbb{Z}, \quad b \neq 0.$$

$$3^b = 2^a.$$

Since 2 and 3 are distinct primes, the given result implies $a = b = 0$. But $b = 0$ is not permitted in a/b, a contradiction.

$$\text{There are no integers } a, b, \text{ with } b \neq 0, \text{ for which } 3 = 2^{a/b}.$$

(2)

(b) A number is rational precisely when it can be written as a/b for integers a, b, with $b \neq 0$. This assumption can be converted using the definition of a logarithm, and part (a) then gives the contradiction.

$$\textit{If } \log_2 3 \in \mathbb{Q}, \textit{ then } \log_2 3 = \frac{a}{b} \quad \textit{for some } a, b \in \mathbb{Z}, \ b \neq 0.$$

$$2^{a/b} = 3 \Rightarrow 2^a = 3^b \Rightarrow 3^b = 2^a,$$

which contradicts part (a).

$$\log_2 3 \notin \mathbb{Q}.$$

(2)