Maths with Matt

Physics and maths resources

Edexcel OCR MEI A Level Further Mathematics

H645/Y435 - Extra Pure

Friday 20 June 2025

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2025 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

For real a, b, define $a \circ b = 3ab$.

(a) Prove that $\circ$ is associative over $\mathbb{R}$. [2]

(b) Determine the identity element e. [2]

(c) Explain why $\mathbb{R}$ is closed under $\circ$. [1]

(d) Explain why $\mathbb{R}$ under $\circ$ is not a group. [2]

Model solution

(a) Associativity is tested by comparing the two bracketings for arbitrary real numbers.

$$(a \circ b) \circ c = (3ab) \circ c = 3(3ab)c = 9abc, \qquad a \circ (b \circ c) = a \circ (3bc) = 3a(3bc) = 9abc.$$

The two expressions are equal for all $a, b, c \in \mathbb{R}$.

$$\circ \text{is associative on} \mathbb{R}.$$

(2)

(b) An identity must leave every a *unchanged. Taking the non-zero element* $a = 1$ *avoids an uninformative zero equation.*

$$1 \circ e = 1 \implies 3e = 1 \implies e = \frac{1}{3}, \qquad a \circ \frac{1}{3} = a \text{ for every } a \in \mathbb{R}.$$

$$e = \tfrac{1}{3}.$$

(2)

(c) If $a, b \in \mathbb{R}$, *then* $a \circ b = 3ab \in \mathbb{R}$, *because a product of real numbers is real.*

$$\mathbb{R} \text{ is closed under } \circ.$$

(1)

(d) A group needs an inverse for every element. For any real b,

$$0 \circ b = 3(0)b = 0 \neq \frac{1}{3} = e.$$

Thus 0 *has no inverse with respect to* $\circ$, *so the inverse axiom fails.*

$$(\mathbb{R}, \circ) \text{ is not a group.}$$

(2)

Question 4

Question 4

The surface S is $z = f(x, y)$, where

$$f(x, y) = x^3 - 12xy^2 + 96y^2 + 30.$$

The point $(0, 0, 30)$ is stationary.

(a) Find all other stationary points. [6]

(b) Use the section $z = f(x, 0)$ to determine the nature of $(0, 0, 30)$. [2]

(c) The normal at $x = 3$, $y = 2$ passes through $(a, 482, 295)$. Find a. [5]

(d) On the curved part C of the contour $z = 542$, there are two points with $x = 8$. Find their y-coordinates. [3]

Model solution

(a) A stationary point of $z = f(x, y)$ has both first partial derivatives equal to zero.

$$f_x = 3x^2 - 12y^2, \qquad f_y = -24xy + 192y.$$

$$f_x = 0, \qquad f_y = 0.$$

The second equation is $y(-24x + 192) = 0$. The case $y = 0$ gives the known point, so an additional point has $x = 8$. Then

$$3(8)^2 - 12y^2 = 0 \implies y^2 = 16 \implies y = \pm 4.$$

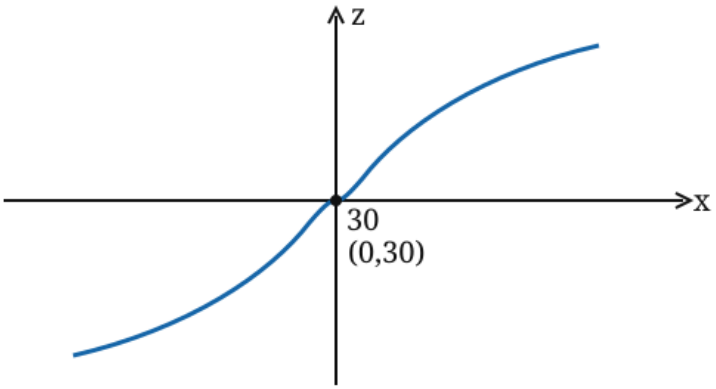
$$f(8, 4) = f(8, -4) = 8^3 - 12(8)(4^2) + 96(4^2) + 30 = 542.$$

The other stationary points are $(8, 4, 542)$ and $(8, -4, 542)$.

(6)

(b) The requested section fixes $y = 0$, so it is the cubic

$$z = f(x, 0) = x^3 + 30.$$



It has a horizontal point of inflection at $(0,30)$*. Moving through it, the section is below* 30 *on one side and above* 30 *on the other, so the surface is neither a local maximum nor a local minimum.*

$$(0,0,30)\text{isasaddlepoint.}$$

(2)

(c) *For* $z = f(x,y)$*, a normal direction is* $(f_x, f_y, -1)$*. Evaluate the point and the partial derivatives at* $(3,2)$*.*

$$f(3,2) = 3^3 - 12(3)(2^2) + 96(2^2) + 30 = 297,$$

$$f_x(3,2) = 3(3)^2 - 12(2)^2 = -21, \qquad f_y(3,2) = -24(3)(2) + 192(2) = 240.$$

$$\mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ 297 \end{pmatrix} + \lambda \begin{pmatrix} -21 \\ 240 \\ -1 \end{pmatrix}.$$

Use the known y*-coordinate of the point on the normal to obtain the parameter, then use its* x*-coordinate.*

$$2 + 240\lambda = 482 \implies \lambda = 2, \qquad a = 3 - 21(2) = -39.$$

$$a = -39.$$

(5)

(d) *Substitute the contour value and rearrange so that the factor associated with* $x = 8$ *is visible.*

$$x^3 - 12xy^2 + 96y^2 + 30 = 542 \implies 512 - x^3 = 12(8-x)y^2.$$

$$512 - x^3 = (8-x)(x^2 + 8x + 64).$$

The curved branch is obtained after cancelling the factor $8 - x$*; the uncancelled factor represents the straight-line component.*

$$y^2 = \frac{x^2 + 8x + 64}{12}.$$

$$x = 8 \implies y^2 = \frac{64 + 64 + 64}{12} = 16.$$

$$y = 4 \text{or} y = -4.$$

(3)

Question 5

Question 5

$$A = \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad a^2 + b^2 = 1, \qquad a \neq \pm 1.$$

(a) (i) Find the characteristic equation in terms of a and show that 1 is always an eigenvalue. [3]

(ii) Show that the other eigenvalues are $a \pm \sqrt{a^2 - 1}$, and prove they are not 1. [3]

(b) Hence show that either some eigenvalues or some entries of A are non-real. [3]

(c) Now a, b are real, A is a rotation through an acute angle in three dimensions, and $\mathbf{e}$ is a real eigenvector for eigenvalue 1.

(i) Explain the significance of $\mathbf{e}$ and its eigenvalue. [2]

(ii) Explain geometrically why the other eigenvectors cannot be real. [1]

(iii) Find a vector equation for the axis of rotation. [2]

Model solution

(a)(i) Eigenvalues satisfy $\det(A - \lambda I) = 0$. *Expanding along the third row, then using* $a^2 + b^2 = 1$, *gives*

$$\det(A - \lambda I) = (1 - \lambda)\big((a - \lambda)^2 + b^2\big) = 0,$$

$$(1 - \lambda)(\lambda^2 - 2a\lambda + a^2 + b^2) = 0 \quad \Longrightarrow \quad (1 - \lambda)(\lambda^2 - 2a\lambda + 1) = 0.$$

The factor $1 - \lambda$ *occurs for every allowable* a.

$$(1 - \lambda)(\lambda^2 - 2a\lambda + 1) = 0; \text{hence} \lambda = 1 \text{is always an eigenvalue.}$$

(3)

(a)(ii) The other eigenvalues are the roots of the quadratic factor.

$$\lambda = \frac{2a \pm \sqrt{4a^2 - 4}}{2} = a \pm \sqrt{a^2 - 1}.$$

If either root were 1, *the quadratic equation would give*

$$1 - 2a + 1 = 0 \quad \Longrightarrow \quad a = 1,$$

contradicting the given condition $a \neq \pm 1$.

$$\lambda = a \pm \sqrt{a^2 - 1}, \text{and neither is} 1.$$

(3)

(b) If a *is non-real,* A *has a non-real entry. Suppose* a *is real. If* $a^2 > 1$, *then* $b^2 = 1 - a^2 < 0$, *so* b *is non-real and* A *again has a non-real entry. If* $0 \leqslant a^2 < 1$, *then* $a^2 - 1 < 0$, *so the two eigenvalues* $a \pm \sqrt{a^2 - 1}$ *are non-real. The remaining possibility* $a^2 = 1$ *is excluded by* $a \neq \pm 1$.

Either some eigenvalues or some entries of A must be non-real.

(3)

(c)(i) A rotation fixes every vector parallel to its axis. Since $A\mathbf{e} = 1\mathbf{e} = \mathbf{e}$, the direction of $\mathbf{e}$ is unchanged, so $\mathbf{e}$ is parallel to the rotation axis. Conversely, the axis consists of invariant vectors, which is why its eigenvalue is 1.

$\mathbf{e}$ gives the direction of the axis of rotation.

(2)

(c)(ii) For an acute rotation the axis is the only invariant direction. A further real eigenvector would give another real invariant direction, which is geometrically impossible for this rotation.

The other eigenvectors cannot be real.

(1)

(c)(iii) Solve $(A - I)\mathbf{e} = \mathbf{0}$. Since the other eigenvalues are not 1, the upper 2×2 equations have only the zero solution, leaving the z-direction.

$$\mathbf{e} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The axis is therefore the line through the origin in this direction.

$$\mathbf{r} = \mu \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mu \in \mathbb{R}.$$

(2)

Question 2

Question 2

$$25u_{n+2} - 35u_{n+1} + 12u_n = c, \qquad n \geqslant 0,$$

where c is constant.

(a) Determine the general solution. [5]

(b) $u_0 = 20$, $u_1 = 70$, and initially $c = -10$.

(i) Determine u_n. [3]

(ii) Find by direct calculation the largest predicted number of containers. [2]

(iii) Show that this model breaks down in the 19th month. [3]

(c) If instead $c = 10$, show that eventually u_n is a constant k, and find k. [2]

Model solution

(a) For the homogeneous recurrence, constant coefficients suggest the trial $u_n = r^n$.

$$25r^2 - 35r + 12 = 0 \quad \Longrightarrow \quad (5r-3)(5r-4) = 0,$$

$$u_n = A\left(\frac{3}{5}\right)^n + B\left(\frac{4}{5}\right)^n.$$

The forcing term is constant, so use the constant trial $u_n = \alpha$.

$$25\alpha - 35\alpha + 12\alpha = c \quad \Longrightarrow \quad \alpha = \frac{c}{2}.$$

$$u_n = A\left(\frac{3}{5}\right)^n + B\left(\frac{4}{5}\right)^n + \frac{c}{2}.$$

(5)

(b)(i) With $c = -10$, the constant term is -5. Substitute the two initial values to find the two unknown coefficients.

$$A + B - 5 = 20 \quad \Longrightarrow \quad A + B = 25,$$

$$\frac{3}{5}A + \frac{4}{5}B - 5 = 70 \quad \Longrightarrow \quad 3A + 4B = 375.$$

$$A = -275, \qquad B = 300.$$

$$u_n = -275\left(\frac{3}{5}\right)^n + 300\left(\frac{4}{5}\right)^n - 5.$$

(3)

(b)(ii) The question asks for direct calculation, so use the recurrence to obtain successive values.

$$u_2 = \frac{35(70) - 12(20) - 10}{25} = 88,$$

$$u_3 = \frac{35(88) - 12(70) - 10}{25} = 89.2, \qquad u_4 = \frac{35(89.2) - 12(88) - 10}{25} = 82.24.$$

The values increase to u_3 and then decrease. The model rounds a number of containers to the nearest integer.

$$89 \text{containers, attheendofmonth} 3.$$

(2)

(b)(iii) Check the end of months 18 and 19 using the explicit formula.

$$u_{18} = 0.376391\ldots, \qquad u_{19} = -0.6933\ldots.$$

The model is non-negative at the end of month 18 but predicts a negative number of containers at the end of month 19. Negative holdings are impossible.

The model breaks down in the 19th month.

(3)

(c) With $c = 10$, the general form becomes

$$u_n = A\left(\frac{3}{5}\right)^n + B\left(\frac{4}{5}\right)^n + 5.$$

Since both bases have magnitude below 1, their powers tend to zero as $n \to \infty$.

$$\lim_{n\to\infty} u_n = 5.$$

$$k = 5.$$

(2)

Question 3

Question 3

$M = (\{0,1,2,3,4,5,6,7,8\}, +_9)$ is the group under addition modulo 9.

(a) State and justify the identity. [1]

(b) Explain why there is no subgroup of order 5. [2]

(c) Find all generators of M. [2]

(d) The only proper non-trivial subgroup is $(H, +_9)$.

(i) Write down H. [1]

(ii) Show there can be no other proper non-trivial subgroup. [2]

Model solution

(a) The identity leaves every residue unchanged under the stated operation.

$$0 +_9 a \equiv a \pmod 9 \qquad (a \in M).$$

The identity is 0.

(1)

(b) The order of M is 9. By Lagrange's theorem, a subgroup order must divide the group order, but $5 \nmid 9$.

No subgroup of M has order 5.

(2)

(c) A residue generates this cyclic group precisely when it is coprime to 9.

The generators are $1, 2, 4, 5, 7, 8$.

(2)

(d)(i) Adding 3 repeatedly modulo 9 produces $0, 3, 6$ and then returns to 0.

$H = \{0, 3, 6\}$.

(1)

(d)(ii) A proper non-trivial subgroup has order dividing 9, but not order 1 or 9; it must have order 3. Every element other than $0, 3, 6$ has order 9, so a subgroup containing one would be all of M, not proper. Hence it can contain only $0, 3, 6$, and an order-3 subgroup is exactly H.

There is no other proper non-trivial subgroup.

(2)