Maths with Matt

Physics and maths resources

Edexcel OCR MEI A Level Further Mathematics

H645/Y436 - Further Pure with Technology

Monday 24 June 2024

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2024 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

Question wording is reproduced from the user-provided 2024 paper for study/reference. Pearson retains copyright in the original examination material.

Contents

Question 1 2

Question 3 4

Question 2 7

www.mathswithmatt.co.uk

Question 1

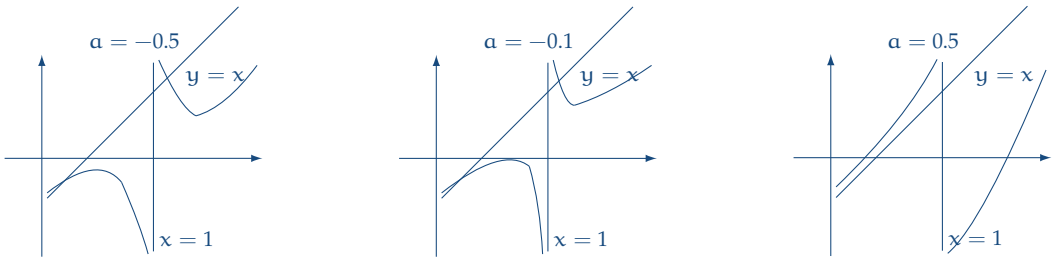
Question 1

For $y = (x^2 - x + a^2 - a)/(x - 1)$, answer (a)(i)–(vii): sketch for $a = -0.5, -0.1, 0.5$; compare their features; find the non-zero value giving a line; give the vertical asymptote; show in $y = x + A + (a^2 - a)/(x - 1)$ that $A = 0$; evaluate $\lim_{x\to\infty}(y - x)$; interpret it. For $0 < a < 1$, the intercepts X, Y and $Z = (0, -1)$ form triangle XYZ: find its area and its maximum. **(17)**

Model solution

Division exposes the geometry: $y = x + \dfrac{k}{x-1}$ *where* $k = a^2 - a$. *Thus* $x = 1$ *and* $y = x$ *are respectively the vertical and oblique asymptotes. Also* $y' = 1 - k/(x-1)^2$.

(a)(i) For $a = -0.5$, $k = 0.75$: *turning points are* $(1-\sqrt{0.75}, 1-2\sqrt{0.75})$ *and* $(1+\sqrt{0.75}, 1+2\sqrt{0.75})$. *For* $a = -0.1$, *they are* $(1-\sqrt{0.11}, 1-2\sqrt{0.11})$ *and* $(1+\sqrt{0.11}, 1+2\sqrt{0.11})$. *In both cases the left branch tends to* $-\infty$ *as* $x \to 1^-$ *and the right branch tends to* $+\infty$ *as* $x \to 1^+$. *For* $a = 0.5$, $k = -0.25$, *so there are no turning points; its left branch tends to* $+\infty$ *as* $x \to 1^-$ *and its right branch to* $-\infty$ *as* $x \to 1^+$. *All branches approach* $y = x$ *at infinity.*



The three sketches are shown above.

(3)

(a)(ii) Only $k > 0$ *permits* $y' = 0$.

Thetwonegative − acurveshaveturningpoints; thea = 0.5curvedoesnot.

(1)

(a)(iii) The remainder must be zero for the curve to equal $y = x$. $a(a-1) = 0$, *so*

$$a = 1.$$

(1)

(a)(iv)

$$x = 1.$$

(1)

(a)(v) Multiply the proposed form by $x - 1$ *and compare coefficients:*

$$(x + A)(x - 1) + a^2 - a = x^2 + (A - 1)x - A + a^2 - a = x^2 - x + a^2 - a.$$

Hence $A - 1 = -1$, *so*

$$A = 0.$$

(2)

(a)(vi) Cancel the quotient before taking the limit.

$$\lim_{x\to\infty}(y-x) = \lim_{x\to\infty}\frac{a^2-a}{x-1} = 0.$$

$$0.$$

(2)

(a)(vii)

$$y = x \text{ is an oblique asymptote.}$$

(1)

(b)(i) The horizontal base is found from the numerator, and Z is one unit below the x-axis.

$$x^2 - x + a^2 - a = 0 \quad \Rightarrow \quad x_X, x_Y = \frac{1 \mp \sqrt{1+4a-4a^2}}{2}.$$

Thus $XY = \sqrt{1+4a-4a^2}$ *and*

$$\text{Area } XYZ = \tfrac{1}{2}\sqrt{1+4a-4a^2}.$$

(4) (b)(ii) Complete the square to see the maximum directly.

$$\frac{1}{2}\sqrt{1+4a-4a^2} = \sqrt{\frac{1}{2} - (a-\frac{1}{2})^2} \leqslant \sqrt{\frac{1}{2}}.$$

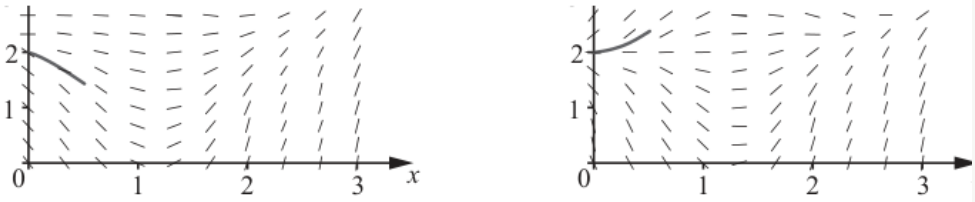
$$\text{Maximum area} = \sqrt{2}/2, \text{ at } a = 1/2.$$

(2)

Question 3

Question 3

For $y' = x^2 - y + a\cos x \cos y$, $x \geqslant 0, y > 0$: for $a = 0$ solve with $y(0) = 1$, find the isoclines and sketch the tangent field. Continue the supplied solution curves through $(0, 2)$, compare them and estimate a for Fig. 3.2. Construct Euler and modified-Euler spreadsheets, use them as specified, and locate the stated turning point. **(20)**



(i) In each case, continue the sketch of the solution curve for $\frac{1}{2} \leqslant x \leqslant 3$ on the axes in th Printed Answer Booklet.

(ii) State **one** feature which is present in the continued solution curve for **Fig. 3.1** that is a feature of the continued solution curve for **Fig. 3.2**.

(iii) Using a slider for a, or otherwise, estimate the value of a for the solution curve show **Fig. 3.2**.

Model solution

(a)(i) For $a = 0$, use the integrating factor e^x for $y' + y = x^2$. $\frac{d}{dx}(e^x y) = x^2 e^x$, hence

$$y = x^2 - 2x + 2 - e^{-x}.$$

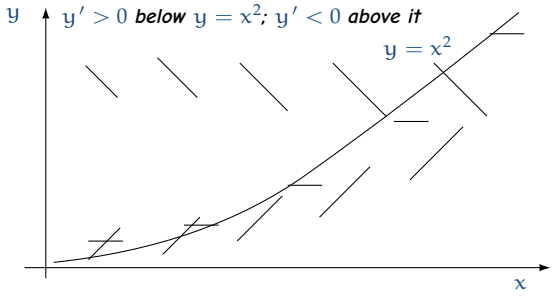
(1)

(a)(ii) Set the gradient equal to the fixed value m. $m = x^2 - y$, so

$$y = x^2 - m, \text{a parabola.}$$

(1)

(a)(iii) Draw constant-slope segments along $y = x^2 - m$; the zero-slope parabola is $y = x^2$, with rising segments below and falling segments above.



(2)

(b)(i) Follow the field, turning where its segments are horizontal.

Fig. 3.1

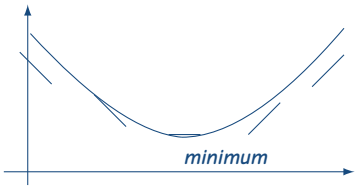


Fig. 3.2

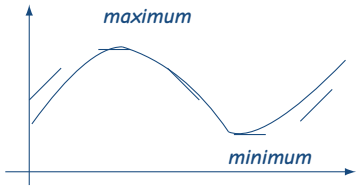


Fig. 3.1 fall to a minimum near $x = 1$ then rises; Fig. 3.2 rises to a maximum near $x = 1$, fall to a minimum, then rises.

(2)

(b)(ii)

Fig. 3.1 has a local minimum close to $x = 1$; Fig. 3.2 does not have a local minimum close to $x = 1$.

(1)

(b)(iii)

$$a \approx -5.5.$$

(1)

(c)(i) Store h *in I1 and* a *in K1. With A2=0, B2=1, Euler's update can be copied down using* $C2 = A2^2 - B2 + \$K\$1 * COS(A2) * COS(B2)$, $D2 = \$I\$1 * C2$, $A3 = A2 + \$I\$1$, $B3 = B2 + D2$.

These formulas give the variable $-$ a, variable $-$ h Euler spreadsheet.

(2)

(c)(ii)

$$y(0.5) \approx 0.618559, \text{ i.e. } 0.6186.$$

(1)

(c)(iii) The exact value from (a)(i) is $0.64346934\ldots$*, an error of* 0.0249.

Correct to 1 decimal place.

(1)

(c)(iv)

Use a smaller h.

(1)

(d)(i) *Modified Euler averages the initial and predicted-endpoint increments. Use* $k_1 = hf(A2, B2)$, $k_2 = hf(A2 + h, B2 + k_1)$, *then* $A3 = A2 + h$, $B3 = B2 + (k_1 + k_2)/2$, *with* $f(x, y) = x^2 - y + a\cos x \cos y$.

These are the modified-Euler spreadsheet formulas.

(2)

(d)(ii)

$y(0.5) \approx 0.504226904$, i.e. 0.5042.

(1)

(d)(iii) *Locate* $y' = 0$ *from the gradient column. At* $h = 0.1$ *the sign changes between* 0.8 *and* 0.9*; at* $h = 0.01$, $0.83 < c < 0.84$.

$c = 0.8$ to 1 decimal place.

(2)

(d)(iv) *The fine-step sheet gives* $0.392182910 < d < 0.392334706$.

$d = 0.392$ to 3 decimal places.

(2)

Question 2

Question 2

Wilson's theorem says $n > 1$ is prime iff $(n-1)! \equiv -1 \pmod{n}$. (a) Evaluate $136! \pmod{137}$ and determine whether 137 is prime. (b) Program Wilson's test up to n, and list primes from 260 to 300. (c) Answer the stated questions about prime residues modulo 4, taking $N = 4c_1c_2c_3 + 3$. (d) Given an assumed finite list $p_1 = 3, \ldots, p_n$ of primes $3 \pmod 4$, prove using $M = 4p_2 \cdots p_n + 3$ that the list is impossible. **(22)**

Model solution

(a)(i)

$$136! \equiv 136 \pmod{137}.$$

(1)

(a)(ii) This is $-1 \pmod{137}$*, so Wilson's iff criterion applies.*

$$137 \text{is prime.}$$

(1)

(b)(i) Test each integer $j \geqslant 2$*; the expression is divisible by* j *exactly for primes.*

```
import math
def wilson_primes(n):
    for j in range(2,n+1):
        if (math.factorial(j-1)+1) % j == 0:
            print(j)
```

$$\text{Thisoutputseveryprimenotexceedingn.}$$

(3)

(b)(ii)

$$263, 269, 271, 277, 281, 283, 293.$$

(1)

(c)(i) A $2 \pmod 4$ *number is even.*

$$2\text{istheonlysuchprime.}$$

(1)

(c)(ii) A $0 \pmod 4$ *number is divisible by* 4*.*

None is prime.

(1)

(c)(iii)

$$c_1 = 263, c_2 = 271, c_3 = 283.$$

(1)

(c)(iv) *Each* c_i *divides the product term.* $N \equiv 3 \pmod{c_i}$, ***so***

$$N \text{ is divisible by none of } c_1, c_2, c_3.$$

(1)

(c)(v) $N = 4c_1c_2c_3 + 3 \equiv 3 \pmod 4$, ***so***

$$3.$$

(1)

(d)(i) $M = 4p_2 \cdots p_n + 3 \equiv 3 \pmod 4$, ***so***

$$3.$$

(1)

(d)(ii) M *is positive, so the fundamental theorem of arithmetic gives a product of primes, with repetitions if necessary.*

$$M = q_1 \cdots q_r, \text{ each } q_i \text{ prime.}$$

(2)

(d)(iii) M *is odd, so its prime factors are* 1 *or* $3 \pmod 4$. *Suppose every* $q_i \equiv 1 \pmod 4$. *Then* $M = q_1 \cdots q_r \equiv 1 \pmod 4$, *contradicting* $M \equiv 3 \pmod 4$.

$$\text{Some } q_i \equiv 3 \pmod 4.$$

(4)

(d)(iv) *By the assumed complete list, the* $q_i \equiv 3 \pmod 4$ *found in (d)(iii) is one of the* p_j. *Each* p_j *with* $j \geqslant 2$ *is non-zero modulo* 3, *so* $M \equiv p_2 \cdots p_n \not\equiv 0 \pmod 3$. *Thus* $3 \nmid M$, *and* q_i *cannot be* $p_1 = 3$; *it is* p_j *for some* $j \geqslant 2$. *Then* $p_j \mid 4p_2 \cdots p_n$, *so* $M \equiv 3 \pmod{p_j}$, *whereas* $q_i = p_j \mid M$ *requires* $M \equiv 0 \pmod{p_j}$. *This contradiction disproves the finite-list assumption.*

$$\text{There are infinitely many primes } 3 \pmod 4.$$

(5)