Maths with Matt

Physics and maths resources

Edexcel OCR MEI A Level Further Mathematics

H645/Y436 - Further Pure with Technology

Monday 23 June 2025

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2025 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

A family of curves is defined by

$$\frac{x^2}{a^2} + mxy + \frac{y^2}{b^2} = 1,$$

where a and b are non-zero real numbers.

(a) Take $a = 2$ and $b = 1$. Sketch the curve for $m = 0$, $m = 1$ and $m = 2$, and state a feature of the $m = 0$ curve which the other two do not have. **(4)**

For the rest of the question $m = 0$.

(b) Verify that $x = a\cos t$, $y = b\sin t$, $0 \leqslant t < 2\pi$, are parametric equations of the curve. **(1)**

(c) Show that $\frac{dy}{dx} = -\frac{b}{a}\cot t$. **(2)**

(d) Show that the normal at parameter t has equation

$$y = \frac{a}{b}\tan t\, x - \frac{a^2 - b^2}{b}\sin t.$$

(5)

(e) Show that the envelope of these normals has parametric equations

$$x = \frac{a^2 - b^2}{a}\cos^3 t, \qquad y = \frac{b^2 - a^2}{b}\sin^3 t.$$

(6)

(f) For $a = 2, b = 1$, sketch this envelope. (g) Find a Cartesian equation of it. **(3)**

Model solution

(a)(i) Substitute the given values first. The resulting quadratic is especially easy to classify for each value of m.

$$\frac{x^2}{4} + mxy + y^2 = 1.$$

For $m = 0$, this is the standard ellipse $x^2/2^2 + y^2/1^2 = 1$: its intercepts are $(\pm 2, 0)$ and $(0, \pm 1)$.

$$m = 1: \quad \frac{x^2}{4} + xy + y^2 = \left(\frac{x}{2} + y\right)^2 = 1,$$

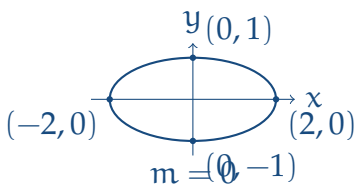
so this sketch is the two parallel lines $y = -x/2 \pm 1$, again meeting the axes at $(\pm 2, 0)$ and $(0, \pm 1)$.

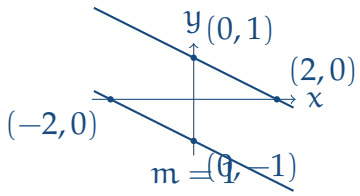
$$m = 2: \quad \frac{x^2}{4} + 2xy + y^2 = 1 \quad \Longleftrightarrow \quad (x + y)^2 - \frac{3}{4}x^2 = 1.$$

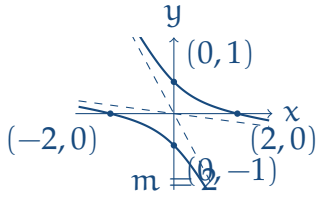
This is a rotated hyperbola. It has the same four intercepts, and its asymptotes are found by setting the homogeneous expression to zero:

$$y = \left(-1 \pm \frac{\sqrt{3}}{2}\right)x.$$

The three labelled sketches are:







Them = 0curveisbounded(equivalently, itisaclosedellipse); neitheroftheothercurvesis

(4)

(b) To verify a parametrisation, substitute it into the Cartesian equation and use $\sin^2 t + \cos^2 t = 1$.

$$\frac{(a\cos t)^2}{a^2} + \frac{(b\sin t)^2}{b^2} = \cos^2 t + \sin^2 t = 1.$$

Hencex = a cos t, y = b sin tparametrisesthecurve.

(1)

(c) As both coordinates are functions of t, *use* $dy/dx = (dy/dt)/(dx/dt)$.

$$\frac{dx}{dt} = -a\sin t, \qquad \frac{dy}{dt} = b\cos t,$$

$$\frac{dy}{dx} = \frac{b\cos t}{-a\sin t} = -\frac{b}{a}\cot t.$$

$$\frac{dy}{dx} = -\frac{b}{a}\cot t.$$

(2)

(d) A normal has gradient the negative reciprocal of the tangent gradient. Then use the point $(a\cos t, b\sin t)$ *on the curve.*

$$m_{\text{normal}} = -\frac{1}{dy/dx} = \frac{a}{b}\tan t,$$

$$y - b\sin t = \frac{a}{b}\tan t\,(x - a\cos t).$$

Expanding the right-hand side puts the line into the required form.

$$y = \frac{a}{b}\tan t\, x - \frac{a^2}{b}\sin t + b\sin t = \frac{a}{b}\tan t\, x - \frac{a^2 - b^2}{b}\sin t.$$

$$y = \frac{a}{b}\tan t\, x - \frac{a^2 - b^2}{b}\sin t.$$

(5)

(e) The envelope of a one-parameter family $f(x,y,t)=0$ is obtained by solving $f=0$ together with $\partial f/\partial t=0$. This identifies the point where neighbouring normals meet. Write the normal as

$$f(x,y,t)=y-\frac{a}{b}\tan t\,x+\frac{a^2-b^2}{b}\sin t=0.$$

$$\frac{\partial f}{\partial t}=-\frac{a}{b}\sec^2 t\,x+\frac{a^2-b^2}{b}\cos t=0.$$

Solving this second equation gives the x-coordinate of the envelope.

$$x=\frac{(a^2-b^2)\cos t}{a\sec^2 t}=\frac{a^2-b^2}{a}\cos^3 t.$$

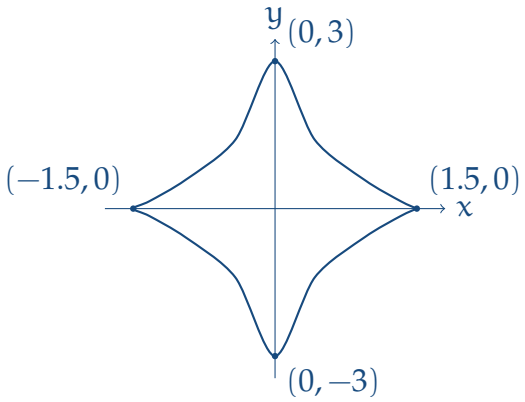
Substitute this into the normal equation to obtain y.

$$\begin{aligned}y&=\frac{a}{b}\tan t\left(\frac{a^2-b^2}{a}\cos^3 t\right)-\frac{a^2-b^2}{b}\sin t\\&=\frac{a^2-b^2}{b}(\cos^2 t-1)\sin t=\frac{b^2-a^2}{b}\sin^3 t.\end{aligned}$$

$$x=\frac{a^2-b^2}{a}\cos^3 t,\quad y=\frac{b^2-a^2}{b}\sin^3 t.$$

(6)

(f) Here the parametrisation becomes $x=\frac{3}{2}\cos^3 t$, $y=-3\sin^3 t$. A sketch is therefore an astroid-shaped four-cusped curve, with cusps at $(\pm1.5,0)$ and $(0,\pm3)$.



The required envelope is the four-cusped astroid shown, through $(\pm1.5,0)$ and $(0,\pm3)$.

(1)

(g) The powers $2/3$ remove the cubes in the parametrisation, so they are the natural choice for eliminating t.

$$(ax)^{2/3}+(by)^{2/3}=(a^2-b^2)^{2/3}\cos^2 t+(a^2-b^2)^{2/3}\sin^2 t.$$

$$(ax)^{2/3}+(by)^{2/3}=(a^2-b^2)^{2/3}.$$

$$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}.$$

(2)

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Question 2

Question 2

(a) (i) Find $7^{10} \pmod{1000}$. (ii) Explain why Fermat's little theorem gives $2^{12} \equiv 1 \pmod{13}$. (iii) Find $2^{12q+1} \pmod{13}$, where q is positive. **(4)**
(b) (i) Find $(354, 27)$. (ii) Write a program returning Euler's totient $\varphi(n)$. (iii) Use it to find $\varphi(1000)$. (iv) Find $7^{400r+10} \pmod{1000}$. (v) Hence find the tens digit of 7^{2010}. **(12)**
(c) If $p \geqslant 3$ is prime and x, y are positive integers, prove that $p^{2x} + 1 = 2^{2y}$ has no integer solutions. **(5)**

Model solution

(a)(i) Only the last three digits are needed, so calculate powers of 7 modulo 1000.

$$7^2 = 49, \qquad 7^4 = 2401 \equiv 401 \pmod{1000},$$

$$7^5 \equiv 7 \cdot 401 = 2807 \equiv 807 \pmod{1000}, \qquad 7^{10} \equiv 807^2 = 651249 \equiv 249 \pmod{1000}.$$

$$7^{10} \equiv 249 \pmod{1000}.$$

(1)

(a)(ii) Fermat's little theorem applies because the modulus 13 is prime and 2 is coprime to 13.

$$2^{13-1} = 2^{12} \equiv 1 \pmod{13}.$$

$$2^{12} \equiv 1 \pmod{13}.$$

(1)

(a)(iii) Split the exponent into blocks of 12, since part (ii) evaluates each such block modulo 13.

$$2^{12q+1} = (2^{12})^q \cdot 2 \equiv 1^q \cdot 2 \equiv 2 \pmod{13}.$$

$$2^{12q+1} \equiv 2 \pmod{13}.$$

(2)

(b)(i) Euclid's algorithm identifies the highest common factor.

$$354 = 13 \cdot 27 + 3, \qquad 27 = 9 \cdot 3.$$

$$(354, 27) = 3.$$

(1)

(b)(ii) The definition says to count the integers from 1 to n whose greatest common divisor with n is 1. A loop and a counter implement this definition directly.

```
import math
def totient(n):
  count = 0
  for m in range(1, n+1):
    if math.gcd(m, n) == 1:
      count = count + 1
  return count
```

The program returns $\varphi(n)$ for every positive integer n.

(4)

(b)(iii) Running the program with $n = 1000$ counts the required coprime integers.

$$\varphi(1000) = 400.$$

(1)

(b)(iv) Euler's theorem is applicable because $(7, 1000) = 1$, and part (iii) supplies the exponent $\varphi(1000) = 400$. Split the power so that the factor 7^{400} can be replaced by 1.

$$7^{400} \equiv 1 \pmod{1000},$$

$$7^{400r+10} = (7^{400})^r 7^{10} \equiv 7^{10} \pmod{1000}.$$

$$7^{10} = 282475249 \equiv 249 \pmod{1000}.$$

$$7^{400r+10} \equiv 249 \pmod{1000}.$$

(4)

(b)(v) The exponent has exactly the form used in part (iv): $2010 = 400 \cdot 5 + 10$. The final three digits determine the tens digit.

$$7^{2010} \equiv 249 \pmod{1000}.$$

The tens digit is 4.

(2)

(c) Reduce the equation modulo 4, because the right-hand side contains 2^{2y}. Since $p \geqslant 3$ is prime, p is odd, so $p^2 \equiv 1 \pmod 4$.

$$p^{2x} + 1 = (p^2)^x + 1 \equiv 1^x + 1 \equiv 2 \pmod 4.$$

But $y \geqslant 1$, so the right-hand side is divisible by 4:

$$2^{2y} = 4^y \equiv 0 \pmod 4.$$

The two sides would have to be congruent modulo 4*, but* $2 \not\equiv 0 \pmod 4$*. This contradiction rules out every possible positive* x, y*.*

There are no integer solutions with $p \geqslant 3$ prime and x, y positive.

(5)

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Question 3

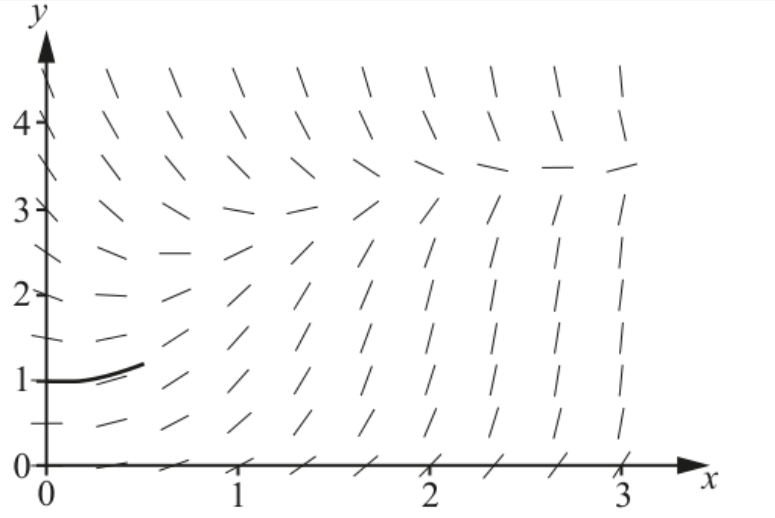
Question 3

Consider

$$\frac{dy}{dx} = x - y + ae^x \sin y.$$

(a) For $a = 0$, verify that $y = 2e^{-x} + x - 1$ satisfies $y(0) = 1$; find the exact x-coordinate of its minimum; and sketch it for $0 \leqslant x \leqslant 3$. **(7)**

(b) The supplied tangent field and the beginning of the solution through $(0, 1)$ are shown below. Continue the solution for $\frac{1}{2} \leqslant x \leqslant 3$.



(2)

(c) Construct a spreadsheet using fourth-order Runge–Kutta for this differential equation, with variable a and h, starting from $x_0 = 0, y_0 = 4$. Then: (ii) for $a = 1, h = 0.05$, find $y(1)$; (iii) for $a = 0.5$, find the minimum's x-coordinate; (iv) for $h = 0.05$, find a to 1 d.p. for which $y(1) = 2.47$ to 2 d.p. **(9)**

Model solution

(a)(i) Verification requires both the differential equation and the initial condition. With $a = 0$, the right-hand side is $x - y$.

$$y = 2e^{-x} + x - 1, \qquad \frac{dy}{dx} = 1 - 2e^{-x}.$$

$$x - y = x - (2e^{-x} + x - 1) = 1 - 2e^{-x} = \frac{dy}{dx}.$$

$$y(0) = 2e^0 + 0 - 1 = 1.$$

The stated function is the required particular solution.

(3)

(a)(ii) At a minimum the tangent is horizontal, so set the derivative equal to zero.

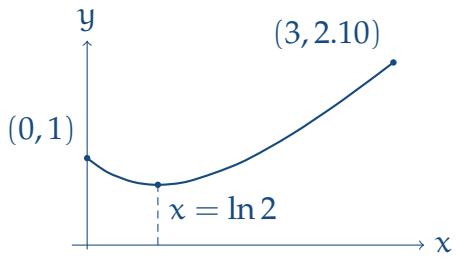
$$1 - 2e^{-x} = 0 \quad \Longrightarrow \quad e^{-x} = \frac{1}{2} \quad \Longrightarrow \quad x = \ln 2.$$

$$m = \ln 2.$$

(2)

(a)(iii) Start at $(0, 1)$. The derivative is initially -1, so the curve falls to its minimum at $x = \ln 2$, then rises. For a useful endpoint check,

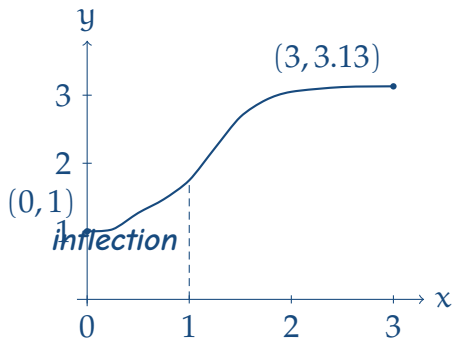
$$y(3) = 2e^{-3} + 2 \approx 2.10.$$



The sketch decreases from $(0, 1)$ to its minimum at $x = \ln 2$, then increases to approximately $(3, 2.10)$, as shown.

(2)

(b) A solution curve must be tangent to the short line segments in the field. Continue the supplied curve from about $(0.5, 1.27)$; it has an inflection at about $x = 1$, then its gradient decreases as it levels off just below $y = \pi$.



The continuation joins the supplied curve, has an inflection near $x = 1$, and follows the tangent − field directions through $(3, 3.13)$.

(2)

(c)(i) Fourth-order Runge--Kutta is appropriate because the question specifies it and it gives a much more accurate step update than Euler's method. Put h in cell N1, a in P1, and use columns A, B for x_n, y_n. Enter $A2 = 0$, $B2 = 4$. With $f(x, y) = x - y + ae^x \sin y$,

use the following formulae in row 2:

cell	*formula*
C2	=A2+$N$1
D2	=A2+$N$1/2
E2	=$N$1*(A2-B2+$P$1*EXP(A2)*SIN(B2))
F2	=B2+E2/2
G2	=$N$1*(D2-F2+$P$1*EXP(D2)*SIN(F2))
H2	=B2+G2/2
I2	=$N$1*(D2-H2+$P$1*EXP(D2)*SIN(H2))
J2	=B2+I2
K2	=$N$1*(C2-J2+$P$1*EXP(C2)*SIN(J2))
A3	=C2
B3	=B2+(E2+2*G2+2*I2+K2)/6

Copy row 3 down: this repeats the RK4 update while a *and* h *remain adjustable.*

The spreadsheet implements the specified RK4 recurrence.

(4)

(c)(ii) Set $a = 1$, $h = 0.05$, *and read the row with* $x = 1$.

$$y(1) = 2.43112005\ldots$$

$$y(1) = 2.43 (\text{to 2 d.p.}).$$

(1)

(c)(iii) Use a small step size, here $h = 0.005$, *so that rows on either side of the minimum can be compared.*

x	y
1.020	2.17905556
1.025	2.17897694
1.030	2.17895237
1.035	2.17898151

The values decrease up to $x = 1.030$ *and increase afterwards, locating the minimum between* 1.025 *and* 1.035.

$$r = 1.03 (\text{to 2 d.p.}).$$

(2)

(c)(iv) Keep $h = 0.05$ *and vary* a, *comparing the value in the* $x = 1$ *row.*

a	$y(1)$
1.00	2.43112005
1.05	2.45159519
1.10	2.47130782
1.15	2.49028155
1.20	2.50854029

The value $a = 1.10$ *gives* $y(1) = 2.47$ *when rounded to two decimal places.*

$$a = 1.1 (to 1 d.p.).$$

(2)

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