Maths with Matt

Physics and maths resources

Edexcel OCR MEI A Level Further Mathematics

H645/Y434 - Numerical Methods

Wednesday 22 June 2022

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2022 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

Let $C = 3.7622$ and $S = 3.6269$ approximate $\cosh 2$ and $\sinh 2$ respectively. Answer parts (a)–(c).

Model solution

(a) Compare further digits of each exact value with the given four-decimal-place approximation. This distinguishes rounding from chopping.

$$\cosh 2 = 3.762195691\ldots, \qquad \sinh 2 = 3.62686041\ldots$$

In each case the next digit is at least 5, so the final retained digit has been increased.

Both approximations are obtained by rounding.

(1)

(b) The exact identity is $\cosh^2 2 - \sinh^2 2 = 1$, so use 1 as the true value when calculating relative error.

$$C^2 - S^2 = 3.7622^2 - 3.6269^2 = 0.99974523\ldots$$

$$\textit{relative error} = \frac{0.99974523\ldots - 1}{1} = -0.00025477\ldots$$

Relative error $= -0.000255$ (3 s.f.).

(2)

(c) No: the calculation is not just a rearrangement performed exactly. Subtracting first and then squaring changes the order of the rounded arithmetic, so its error need not be the same.

The relative error is not necessarily the same.

(1)

Question 2

Question 2

For $x = 2.75, 3, 3.25$, the corresponding values of $f(x)$ are $0.920799, 1, 1.072858$. Also, $f(5) = 1.4645$ and $f'(5) = 0.1820$ (4 d.p.).

Model solution

(a) A forward difference uses the point at $x = 3$ and the next tabulated point, $x = 3.25$.

$$f'(3) \approx \frac{f(3.25) - f(3)}{3.25 - 3} = \frac{1.072858 - 1}{0.25} = 0.291432.$$

0.29143

(2)

(b) A central difference uses equally spaced values on both sides of 3, which gives a more balanced estimate.

$$f'(3) \approx \frac{f(3.25) - f(2.75)}{3.25 - 2.75} = \frac{1.072858 - 0.920799}{0.5} = 0.304118.$$

0.30412

(2)

(c) The central-difference formula is second-order accurate in the step size, whereas the forward-difference formula is only first-order accurate. Hence (b) is expected to be closer to the true derivative.

The central-difference estimate is likely to be closer.

(1)

(d) For a small change Δx, linearisation gives $\Delta f \approx f'(5)\Delta x$. Here $\Delta x = 5.024 - 5 = 0.024$.

$$\Delta f \approx 0.1820(0.024) = 0.004368.$$

$$\text{Theerrorisapproximately} \pm 0.00437 (\text{or} \pm 0.0044).$$

(2)

Question 3

Question 3

For $f(x) = \sin^{-1} x - x + 0.1 = 0$, a spreadsheet lists $x_0 = -1$, $x_1 = -0.8$, $x_2 = -0.787691$, $x_3 = -0.7576546$, $x_4 = -0.7540834$, with their function values. It calculates each next x from the preceding two points by a straight-line intercept formula.

Model solution

(a) Using two previous function values to find the x-axis intercept of their secant is the secant method.

Secant method.

(1)

(b) Apply the same secant-intercept formula to the last two displayed pairs; this replaces the tangent in Newton--Raphson by a secant.

$$x_5 = \frac{(-0.7576546)(-0.0001740) - (-0.7540834)(-0.0020574)}{-0.0001740 - (-0.0020574)} = -0.7537535\ldots$$

Using $(x_4, f(x_4))$ and $(x_5, f(x_5))$ in the next secant step gives

$$x_6 = -0.7537502\ldots$$

$$x_5 = -0.7537535, \quad x_6 = -0.7537502$$

(3)

(c) Two successive approximations agree to five decimal places, so those digits are supported by the iteration.

$$\alpha = -0.75375$$

(1)

(d) The spreadsheet retains more digits internally than it displays. The calculator calculation used the displayed, rounded entries, so it need not reproduce cell D9 exactly.

The spreadsheet used higher-precision values than those displayed.

(1)

(e) The notation E−09 means "multiply by 10^{-9}".

$$-1.4629 \times 10^{-9}$$

(1)

Question 4

Question 4

The equation $e^x - x^2 - x - 1.1 = 0$ has a root α with $1 < \alpha < 2$. Consider Newton–Raphson and the iteration $x_{n+1} = \ln(x_n^2 + x_n + 1.1)$, with starting value 1 or 2.

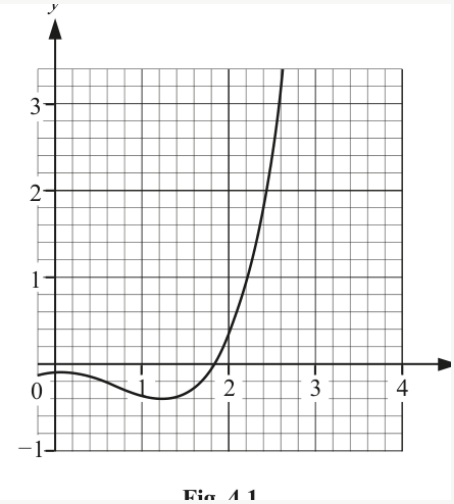
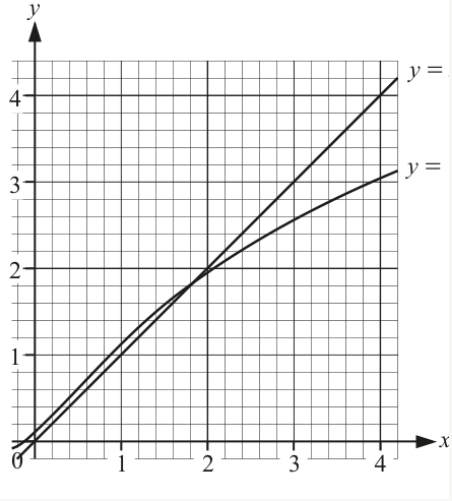


Fig. 4.1

Fig. 4.2 shows parts of the graphs of $y = x$ ar



Model solution

(a) Newton--Raphson follows a tangent to the x-axis. From the graph, the tangent from $x_0 = 1$ meets the axis further from the required root, while the tangent from $x_0 = 2$ meets it closer.

$$\text{Use } x_0 = 2.$$

(2)

(b) For fixed-point iteration, the cobweb diagram shows that either starting value moves towards the intersection. The gradient of $y = \ln(x^2 + x + 1.1)$ at the intersection has magnitude less than 1, which is the convergence condition. Starting at 2 is preferable because it is nearer.

$$\text{Either works; } x_0 = 2 \text{ is the better choice.}$$

(2)

(c)(i) Column N is formed by subtracting successive entries in column M.

$$x_{n+1} - x_n, \text{ the difference between consecutive iterates.}$$

(1)

(c)(ii) Column O divides one consecutive difference by the preceding one.

The ratio of consecutive differences.

(1)

(d) The ratios in column O decrease $(0.172, 0.028, 8 \times 10^{-4}, 6 \times 10^{-7})$, rather than tending to a non-zero constant. This indicates faster-than-first-order convergence, characteristic

of Newton--Raphson rather than fixed-point iteration.

Ali used Newton--Raphson.

(2)

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Question 5

Question 5

For an integral $\int_a^b f(x)\,dx$, a spreadsheet gives $M_1 = 0.2436699$, $T_1 = 0.1479020$ and $M_2 = 0.2306967$, where M_n, T_n, S_{2n} are midpoint, trapezium and Simpson approximations.

Model solution

(a) When the number of strips is doubled, the new trapezium estimate can be obtained by averaging the old midpoint and trapezium estimates.

$$T_2 = \frac{M_1 + T_1}{2}.$$

```
=(G4+H4)/2
```

(2)

(b) Simpson's rule is the weighted combination $S_{2n} = (2M_n + T_n)/3$. First calculate T_2, then apply this formula for $n = 1, 2$.

$$T_2 = \frac{0.2436699 + 0.1479020}{2} = 0.1957860,$$

$$S_2 = \frac{2(0.2436699) + 0.1479020}{3} = 0.2117473, \qquad S_4 = \frac{2(0.2306967) + 0.1957860}{3} = 0.2190598.$$

n	M_n	T_n	S_{2n}
1	0.2436699	0.1479020	0.2117473
2	0.2306967	0.1957860	0.2190598

$$T_2 = 0.1957860, S_2 = 0.2117473, S_4 = 0.2190598.$$

(4)

(c) Simpson's rule has error proportional to h^4. Halving h therefore reduces the error by 16, so Richardson extrapolation removes the leading error.

$$I \approx S_4 + \frac{S_4 - S_2}{16 - 1} = 0.2190598 + \frac{0.2190598 - 0.2117473}{15} = 0.2195473\ldots$$

$$\int_a^b f(x)\,dx \approx 0.2195473 (\text{so} 0.220 \text{to} 3\text{d.p.}).$$

(3)

Question 6

Question 6

Fixed-point iteration $x_{n+1} = g(x_n)$ is used for $\sin^{-1} x - x^2 + 1 = 0$, where $g(x) = \sin(x^2 - 1)$. Late iterations include $x_{107} = -0.600404$, $x_{108} = -0.596806$, difference $d = 0.003598$, and ratio $r = -0.95889$.

Model solution

(a) The late differences have an approximately constant ratio, so treat them as a geometric tail. Adding that tail to the latest iterate gives an improved limit estimate.

$$\alpha \approx x_{108} + d\frac{r}{1-r} = -0.596806 + 0.003598\frac{-0.95889}{1-(-0.95889)} = -0.598567\ldots$$

$$\alpha \approx -0.598568 (\text{hence} - 0.5986 \text{to} 4\text{d.p.}).$$

(4)

(b) Use $x_{n+1} = 0.49x_n + 0.51\sin(x_n^2 - 1)$. The difference is $x_n - x_{n-1}$ and the ratio is the current difference divided by the previous difference.

r	x_r	*difference*	*ratio*
0	0		
1	−0.4291502	−0.429	
2	−0.5817139	−0.153	0.356
3	−0.5983772	−0.0167	0.109
4	−0.5985695	−0.000192	0.0115
5	−0.5985697	-1.99×10^{-7}	0.00103
6	−0.5985697	-1.82×10^{-10}	0.000914

(4)

(c) The final iterates agree to seven decimal places.

$$-0.5985697$$

(1)

(d) Extrapolation requires a constant limiting ratio of differences. Here the ratios are changing, so the required geometric-error model is not valid.

Extrapolation cannot be used.

(1)

(e) Relaxation can also turn a divergent iterative scheme into a convergent one by replacing the update with a weighted average. *(1)*

Question 7

Question 7

Sam's masses (kg) after $t = 0, 10, 20, 30$ days are $88.3, 80.05, 78.7, 78.85$. Later data are $M(40) = 78.7$ and $M(50) = 80.05$.

Model solution

(a) Successive differences use the equally spaced times (spacing 10 days).

t	M	ΔM	$\Delta^2 M$	$\Delta^3 M$
0	88.3	-8.25		
10	80.05	-1.35	6.9	
20	78.7	0.15	1.5	-5.4
30	78.85			

(1)

(b) Four equally spaced data values determine a cubic. Newton's forward interpolation form uses the first value and the first, second and third differences, making the given table especially useful.

$$M = 88.3 - \frac{8.25t}{10} + \frac{6.9t(t-10)}{2(10)^2} - \frac{5.4t(t-10)(t-20)}{6(10)^3}.$$

Expanding gives

$$M = -0.0009t^3 + 0.0615t^2 - 1.35t + 88.3.$$

$$M = -0.0009t^3 + 0.0615t^2 - 1.35t + 88.3.$$

(4)

(c) Test the model on the later observations rather than merely on the four data values used to construct it.

$$M(40) = 75.1, \qquad M(50) = 62.05.$$

These disagree substantially with the observed 78.7 and 80.05.

The cubic is a poor fit for the later data.

(2)

(d) Extend the difference table using the two new observations. A constant fourth difference is the clue that a quartic interpolating model is appropriate.

t	M	ΔM	$\Delta^2 M$	$\Delta^3 M$	$\Delta^4 M$
0	88.3				
10	80.05	-8.25			
20	78.7	-1.35	6.9		
30	78.85	0.15	1.5	-5.4	
40	78.7	-0.15	-0.3	-1.8	3.6
50	80.05	1.35	1.5	1.8	3.6

The fourth differences are constant (3.6), so a quartic may be appropriate.

(2)

(e) *Add the fourth-difference term to the cubic Newton form. Its denominator is* $4!\,10^4$ *because four factors each have time-step scale* 10.

$$M = -0.0009t^3 + 0.0615t^2 - 1.35t + 88.3 + \frac{3.6t(t-10)(t-20)(t-30)}{4!\,10^4}.$$

On expanding,

$$M = 0.000015t^4 - 0.0018t^3 + 0.078t^2 - 1.44t + 88.3.$$

(3)

(f) *The positive quartic coefficient dominates for large positive* t, *so the model predicts that Sam's mass increases without bound as time increases. That is not physically reasonable over a long period.*

It is not appropriate in the long run.

(2)

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