Edexcel OCR MEI A Level Further Mathematics

H645/Y434 - Numerical Methods

Thursday 22 June 2023

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2023 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

Question wording is reproduced from the user-provided 2023 paper for study/reference. Pearson retains copyright in the original examination material.

Contents

Question 1 2

Question 2 3

Question 3 4

Question 4 6

Question 5 7

Question 6 9

Question 7 10

Question 8 11

Question 9 13

Question 1

Question 1

The rounded coordinates are $(x_1, y_1) = (0.9, 2.3)$ and $(x_2, y_2) = (1.1, 2.7)$. Find ranges for the differences and for the gradient $m = (y_2 - y_1)/(x_2 - x_1)$, and explain the size of the resulting uncertainty.

Model solution

(a) A number rounded to one decimal place is within 0.05 of its displayed value. Find the extreme differences by pairing the smallest possible x_2 with the largest possible x_1, and conversely.

$$0.85 < x_1 < 0.95, \qquad 1.05 < x_2 < 1.15.$$

$$1.05 - 0.95 < x_2 - x_1 < 1.15 - 0.85,$$

$$0.1 < x_2 - x_1 < 0.3$$

(2)

(b) Chopping to one decimal place gives a value at least as large as the displayed value and less than 0.1 above it. Apply the same extreme-value idea.

$$2.3 \leqslant y_1 < 2.4, \qquad 2.7 \leqslant y_2 < 2.8.$$

$$2.7 - 2.4 < y_2 - y_1 < 2.8 - 2.3,$$

$$0.3 < y_2 - y_1 < 0.5$$

(2)

(c) Both numerator and denominator are positive, so the smallest gradient uses the smallest numerator and largest denominator; the largest gradient uses the largest numerator and smallest denominator.

$$\frac{0.3}{0.3} < m < \frac{0.5}{0.1}.$$

$$1 < m < 5$$

(2)

(d) The gradient divides by $x_2 - x_1$. This denominator is formed by subtracting two nearly equal quantities, so its relative uncertainty is large and is amplified by the division.

Thesubtractioninthedenominatormakestheuncertaintyinmmuchlarger.

(1)

Question 2

Question 2

A tyre has data $(t, P) = (0, 34.5), (3, 29.4), (5, 27.0)$, where t is in days and P is in psi. A quadratic model is proposed.

Model solution

(a) Newton forward differences require equally spaced values of the independent variable. Here the gaps in time are 3 days and 2 days.

$$\text{Newton's forward} - \text{difference interpolation cannot be used because the} - \text{values are not equally spaced.}$$

(1)

(b) The three data points determine a quadratic. Lagrange interpolation is appropriate because it works directly with unequally spaced data.

$$P(t) = 27.0\frac{(t-0)(t-3)}{(5-0)(5-3)} + 29.4\frac{(t-0)(t-5)}{(3-0)(3-5)} + 34.5\frac{(t-3)(t-5)}{(0-3)(0-5)}.$$

Each Lagrange term is 1 at its own data value and 0 at the other two, so this expression has exactly the required three values. Simplifying gives

$$P = 0.1t^2 - 2t + 34.5$$

(4)

(c) Test the model at the later observed times, rather than assuming that interpolation guarantees a good extrapolation.

$$P(6) = 0.1(6)^2 - 2(6) + 34.5 = 26.1 \approx 26.0,$$

$$P(10) = 0.1(10)^2 - 2(10) + 34.5 = 24.5 \approx 24.4.$$

The model is a good fit to these further data, since both predictions are close to the observed pressures.

(2)

(d) The positive t^2 coefficient means that the parabola eventually turns upwards. A punctured tyre should not begin to gain pressure without intervention.

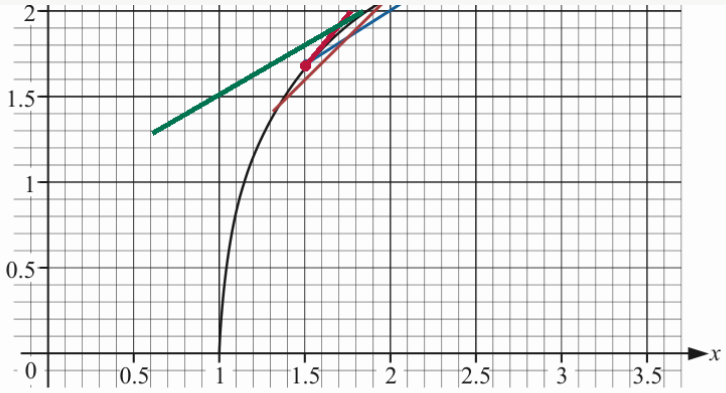
It is unsuitable in the long term because it eventually predicts that the tyre pressure increases.

(1)

Question 3

Question 3

For $f(1.5) = 1.682137$, $f(2) = 2.094395$, and $f(2.5) = 2.318559$, approximate $f'(2)$ by forward and central differences and interpret these on the graph.



The table shows some values of x and the associated values of y.

x	1.5	2	2.5
y	1.682137	2.094395	2.318559

Model solution

(a) A forward difference uses the point at $x = 2$ and the next point to its right.

$$f'(2) \approx \frac{f(2.5) - f(2)}{2.5 - 2} = \frac{2.318559 - 2.094395}{0.5}.$$

$$f'(2) \approx 0.448328$$

(2)

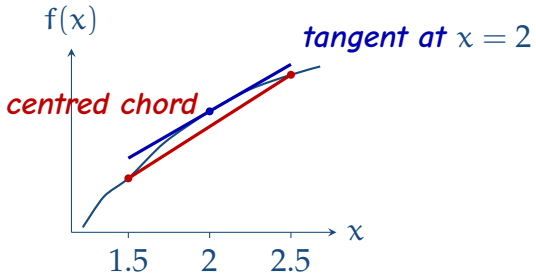
(b) A central difference uses equally distant points on both sides of 2, so its chord is centred at the point where the derivative is required.

$$f'(2) \approx \frac{f(2.5) - f(1.5)}{2.5 - 1.5} = \frac{2.318559 - 1.682137}{1}.$$

$$f'(2) \approx 0.636422$$

(2)

(c) The graph shows the centred chord joining the points on the curve with $x = 1.5$ and $x = 2.5$, together with the tangent at $x = 2$. The gradient of the chord is the approximation to the tangent gradient.



The required construction is the centred chord from $x = 1.5$ to $x = 2.5$, compared with the tangent at $x = 2$.

(1)

(d) *The central difference is normally second-order accurate, while a forward difference is normally first-order accurate. It also samples both sides of* $x = 2$.

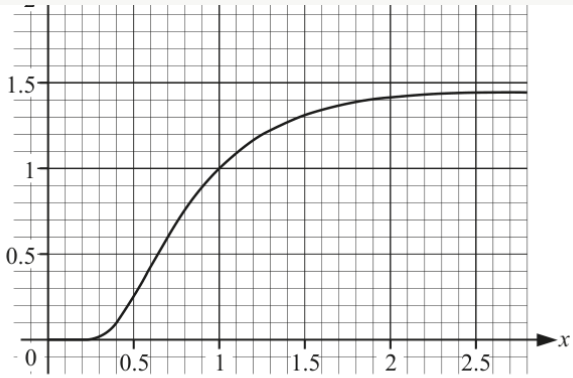
The answer in part (b) is likely to be the better approximation.

(1)

Question 4

Question 4

A one-strip midpoint-rule spreadsheet has midpoint 1.5, function value 1.3103707, and output 0.65518535. The function-value formula is $x^{1/x}$.



(b) Explain whether 0.65518535 is an over-estimate or an under-estimate of $\int_a^b f(x)\,dx$. [1

Model solution

(a) For one midpoint strip, the output is strip width times the function value at the midpoint. Since $0.65518535 = 0.5 \times 1.3103707$, *the width is* 0.5. *The limits must be symmetric about the midpoint* 1.5.

$$\int_{1.25}^{1.75} x^{1/x}\,dx.$$

(3)

(b) The graph is concave down on this interval. A midpoint rectangle for a concave-down curve lies above the curve, so its area is too large.

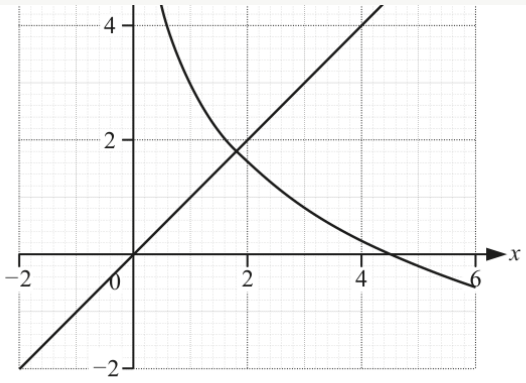
$$0.65518535 \text{is an over} - \text{estimate.}$$

(1)

Question 5

Question 5

The equation $3 - 2\ln x - x = 0$ has a root near 1.8. The proposed fixed-point iteration is $x_{n+1} = g(x_n) = 3 - 2\ln x_n$; a relaxed iteration with $\mu = 0.475$ and $x_0 = 2$ is also considered.



(a) With reference to the graph, explain why it might not be possible to use the student's iterativ formula to find the root near $x = 1.8$. [1

(b) Use the relaxed iteration $x_{n+1} = \lambda g(x_n) + (1-\lambda)x_n$, with $\lambda = 0.475$ and $x_0 = 2$, to determin the root correct to **6** decimal places. [3

Model solution

(a) Fixed-point iteration converges locally only when $|g'(x)| < 1$ near the fixed point. From the graph, the gradient of g at its intersection with $y = x$ is less than -1.

The proposed iteration need not converge, because $|g'(1.8)| > 1$

(1)

(b) Relaxation replaces the next raw fixed-point value by a weighted move from the current value. This is used here to make the iteration converge.

$$x_{n+1} = 0.475\,(3 - 2\ln x_n) + 0.525x_n.$$

n	x_n
0	2.00000000000
1	1.81651017847
2	1.81159652649
3	1.81159008685
4	1.81159008298

The last two values agree to six decimal places, so the requested rounding is secure.

$x = 1.811590$ *(6 d.p.)*

(3)

(c) The ratios of successive differences approach the non-zero constant 0.0006. A constant ratio of errors/differences is the signature of first-order convergence.

The sequence has first-order convergence.

(2)

Question 6

Question 6

Consider using π chopped to two decimal places to approximate π^2+2 and $(\pi+2)^2$. A spreadsheet also calculates 10^x, displayed as 1, followed by $\log_{10}(10^x)$, for exact $x = 10^{-12}$ and 10^{-11}.

Model solution

(a)(i) Chopping π to two decimal places gives 3.14. Relative error is (approximation − exact)/(exact).

$$\frac{(3.14)^2+2-(\pi^2+2)}{\pi^2+2} = -0.000843\ldots$$

$$\textit{relative error} \approx -8.43\times10^{-4}$$

(2)

(a)(ii) No: the two expressions apply the addition and squaring in a different order, so rounding π changes them by different relative amounts.

The relative error would be different.

(1)

(b)(i) The spreadsheet displays the value calculated by $\log_{10}(10^{10^{-12}})$, not merely the rounded display of the intermediate cell.

$$1.00001\times10^{-12}.$$

(1)

(b)(ii) Although cells B2 and B3 display as 1, their stored values are slightly different from 1, and from each other. The logarithm therefore acts on those stored values, not on the display.

Theresultsarenon − zerobecausethestored10^xvaluesarenotexactly1.AlthoughB2andB3bothdispla

(2)

Question 7

Question 7

At $x = 2$, $f(2) = 6$ and $f'(2) = -2.8$. Newton–Raphson iterates include 2.182419024 and 2.182419066, with ratios of consecutive differences decreasing towards zero.

Model solution

(a) For a small change $h = 0.03$*, use the tangent-line approximation* $f(2+h) \approx f(2)+hf'(2)$.

$$f(2.03) \approx 6 + 0.03(-2.8) = 5.916.$$

Thus the estimated value is 0.084 *below* $f(2)$*; the signed error in using* $f(2)$ *as the estimate is* $f(2.03) - f(2)$.

$$\textit{error} \approx -0.084$$

(2)

(b)(i) For first-order convergence, ratios of consecutive differences approach a non-zero constant. Here the ratios decrease rapidly towards zero instead.

The convergence is faster than first order (not first order).

(2)

(b)(ii) The final two displayed approximations agree through seven decimal places; their difference is 4.2×10^{-8}*. The faster-than-first-order convergence supports using the final displayed value to the precision shown.*

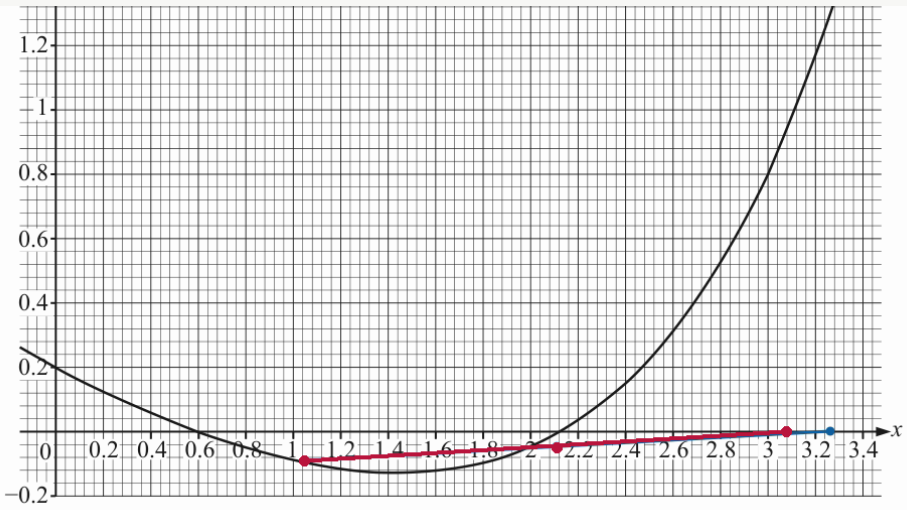
$$a \approx 2.182419066$$

(2)

Question 8

Question 8

For $f(x) = 0.2\cosh x - 0.4x$, the roots $\alpha < \beta$ lie in $(0, 3)$. The secant method starts with $x_0 = 1$, $x_1 = 2$ to find β.

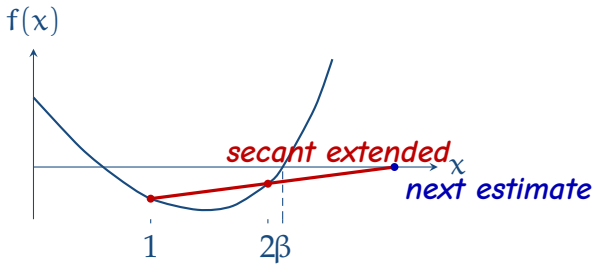


The equation $0.2\cosh x - 0.4x = 0$ has two roots, α and β where $\alpha < \beta$, in the interval $0 < x < 3$. The secant method with $x_0 = 1$ and $x_1 = 2$ is to be used to find β.

(a) On the copy of the graph in the Printed Answer Booklet, show how the secant method works with these two values of x to obtain an improved approximation to β. **[1**

Model solution

(a) The graph shows the secant chord through the curve at $x = 1$ and $x = 2$, extended to its x-axis intercept to the right of the root; this intercept is the next estimate.



Drawthechordthroughthecurveatx = 1andx = 2;itsinterceptwiththex − axis,totherightoftheroot,istheimprovedapproximation.

(1)

(b) Column J holds the current x_r, while column L holds the next x_{r+1}. The next row therefore starts from the previous next estimate.

=L3

(1)

(c) Use the equation of the secant through $(J3, K3)$ *and* $(L3, M3)$*, then put* $y = 0$ *to find its intercept.*

$$x_{r+1} = \frac{x_r f(x_{r+1}) - x_{r+1} f(x_r)}{f(x_{r+1}) - f(x_r)}.$$

```
=(J3*M3-K3*L3)/(M3-K3)
```

(2)

(d) The latest secant estimate shown is the most accurate displayed estimate.

$$\beta \approx 2.12664$$

(1)

(e) To check rounding to five decimal places, test the two endpoints of the rounding interval, without generating another iterate.

$$f(2.126635) \approx -0.00007038, \qquad f(2.126645) \approx -0.00006611.$$

There is no sign change across this interval, so the root is not trapped within the interval that would round to 2.12664*.*

2.12664 ***is not correct to 5 decimal places.***

(2)

(f) To converge to the smaller root α *from* $x_0 = 1$*, choose the second starting value* $a > 1$ *before the turning point of the curve. The graph gives the suitable range.*

$1 < a \leqslant 1.74$ ***(for example,*** $a = 1.5$***).***

(1)

Question 9

Question 9

For $I = \int_0^1 \sqrt[3]{\sinh(x)}\,dx$, trapezium-rule values are $T_1 = 0.52764369$, $T_2 = 0.66617652$, $T_4 = 0.72534275$. Further values include $T_{64} = 0.7655404$, with last difference 0.001618 and ratio about 0.4.

Model solution

(a) Simpson's rule can be obtained by Richardson extrapolation from trapezium estimates whose strip numbers differ by a factor of two: $S_{2n} = (4T_{2n} - T_n)/3$.

$$S_2 = \frac{4T_2 - T_1}{3} = \frac{4(0.66617652) - 0.52764369}{3} = 0.71235413\ldots,$$

$$S_4 = \frac{4T_4 - T_2}{3} = \frac{4(0.72534275) - 0.66617652}{3} = 0.74506483\ldots.$$

$$S_2 \approx 0.712354, \qquad S_4 \approx 0.745065$$

(2)

(b) The two Simpson estimates agree only in their first decimal place, so that is the precision justified by these results.

$$I \approx 0.7 \ \textit{(1 d.p.)}$$

(1)

(c) The ratios of successive trapezium differences approach 0.4. *For halved step size, an order* p *method has ratio* 2^{-p}; *since* $0.25 < 0.4 < 0.5$, $1 < p < 2$.

The method has order between first and second order in this case.

(2)

(d) If the remaining errors form a geometric pattern with ratio $r \approx 0.4$, *extrapolate the final estimate by adding the remaining tail* $d\,r/(1-r)$, *where* $d = 0.001618$ *is the latest difference.*

$$I \approx 0.7655404 + 0.001618\left(\frac{0.4}{1-0.4}\right) = 0.766619066\ldots.$$

Extrapolation is much more accurate than the raw trapezium value, so the digits through four decimal places are justified.

$$I \approx 0.7666 \ \textit{(4 d.p.)}$$

(4)