Maths with Matt

Physics and maths resources

Edexcel OCR MEI A Level Further Mathematics

H645/Y434 - Numerical Methods

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Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2024 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

For $x = 1.9, 2, 2.1$, the corresponding values of $f(x)$ are $0.5842, 0.6309, 0.6753$. (a) Estimate $f'(2)$ as accurately as possible. (b) Estimate the error in using $f(2)$ to estimate $f(2.05)$.

Model solution

(a) The values on either side of 2 are equally spaced, so the central-difference estimate uses both and is the most accurate available estimate.

$$f'(2) \approx \frac{f(2.1) - f(1.9)}{2.1 - 1.9} = \frac{0.6753 - 0.5842}{0.2} = 0.4555.$$

$$f'(2) \approx 0.4555.$$

([2])

(b) A small change $\Delta x = 0.05$ gives $\Delta f \approx f'(2)\Delta x$. This estimates how much the function value changes from $x = 2$ to $x = 2.05$.

$$f(2.05) \approx 0.6309 + 0.05(0.4555) = 0.653675.$$

error in using $f(2) = 0.6309 - 0.653675 = -0.022775.$

$$\text{Theestimateisanunderestimatebyabout0.0228(signederror} - 0.0228).$$

([2])

Question 2

Question 2

Let $a = \tanh(1)$ and $b = \tanh(2)$. A and B are these values rounded to one decimal place. Find the relative errors R_A, R_B, then the relative error R_C in $C = A/B$ as an approximation to $c = a/b$, and comment on their relationship.

Model solution

(a) Relative error compares the size of the error with the exact value: $R = |\text{approximation} - \text{exact}|/|\text{exact}|$. First calculate the unrounded values.

$$a = \tanh(1) = 0.761594\ldots, \quad A = 0.8, \qquad b = \tanh(2) = 0.964028\ldots, \quad B = 1.0.$$

$$R_A = \frac{|0.8 - 0.761594\ldots|}{0.761594\ldots} = 0.0504\ldots, \qquad R_B = \frac{|1.0 - 0.964028\ldots|}{0.964028\ldots} = 0.0373\ldots.$$

$$R_A \approx 0.0504, R_B \approx 0.0373.$$

([3])

(b) Here the exact quotient and its approximation should be compared directly.

$$c = \frac{a}{b} = \frac{0.761594\ldots}{0.964028\ldots} = 0.7900128\ldots, \qquad C = \frac{A}{B} = \frac{0.8}{1.0} = 0.8.$$

$$R_C = \frac{|0.8 - 0.7900128\ldots|}{0.7900128\ldots} = 0.0126\ldots.$$

$$R_C \approx 0.0126.$$

([2])

(c) For a quotient, the signed relative errors approximately subtract; the two rounding errors partly cancel here.

$$R_C \approx R_A - R_B = 0.0504 - 0.0373 = 0.0131 \textit{ (approximately)}.$$

$$R_C \text{is approximately} R_A - R_B (\text{and is smaller than either error}).$$

([1])

Question 3

Question 3

The equation $x^2 - \cosh(x-2) = 0$ has roots $a < 1 < b$. Use $x_{n+1} = g(x_n) = \sqrt{\cosh(x_n - 2)}$, starting at $x_0 = 1$, to find a. Explain why this iteration cannot find b. Then use relaxed iteration $x_{n+1} = (1-\mu)x_n + \mu g(x_n)$ with $\mu = -0.21$, $x_0 = 6.4$, to find b. State another use of relaxation.

Model solution

(a) A fixed point of g satisfies $x = \sqrt{\cosh(x-2)}$, which is exactly the given equation. Repeatedly apply the stated iteration until successive values settle to three decimal places.

$$x_1 = 1.2422079676, \quad x_2 = 1.1406709307, \quad \ldots$$

$$x_9 = 1.168754891, \quad x_{10} = 1.168692142, \quad x_{11} = 1.168717118, \quad x_{12} = 1.168707177.$$

$$a = 1.169 (3\text{d.p.}).$$

([2])

(b) A fixed-point iteration is locally convergent only when the magnitude of the gradient of the iteration function is less than 1. At the larger root, the graph shows that g is steeper than $y = x$.

$$g'(x) = \frac{\sinh(x-2)}{2\sqrt{\cosh(x-2)}}, \qquad g'(b) > 1.$$

$$\text{Since} |g'(b)| > 1, \text{errors grow rather than shrink, so the iteration diverges from } b.$$

([1])

(c) Relaxation replaces g by a weighted combination of x_n and $g(x_n)$; the negative value of μ reduces the effective gradient and makes this fixed point convergent.

$$x_{n+1} = 1.21x_n - 0.21\sqrt{\cosh(x_n - 2)}.$$

$$x_1 = 6.4037529, \quad x_2 = 6.4057774, \quad \ldots, x_8 = 6.4080788, \quad x_9 = 6.4081051, \quad x_{10} = 6.4081192.$$

$$b = 6.408 (3\text{d.p.}).$$

([2])

(d) Relaxation can also improve an iteration that already converges, by making its convergence faster.

For example, it can be used to accelerate the convergence of a convergent sequence.

([1])

Question 4

Question 4

For $t = 0, 1, 2, 3, 4$, temperatures T are $8.8, 13.2, 15.4, 15.4, 13.3$ °C. Complete the difference table; form a quadratic Newton forward interpolating polynomial; assess it against October (8.9°C, $t = 5$) and November (7.5°C, $t = 6$). For 2022, the corresponding May value is 10.3 while the successive differences are unchanged; adapt the model.

Model solution

(a) First differences are consecutive changes in T; second differences are changes in those first differences.

t	T	ΔT	$\Delta^2 T$
0	8.8	4.4	−2.2
1	13.2	2.2	−2.2
2	15.4	0.0	−2.1
3	15.4	−2.1	
4	13.3		

Thefirstdifferencesare$4.4, 2.2, 0, -2.1$andseconddifferencesare$-2.2, -2.2, -2.1$.

([2])

(b) A quadratic has constant second differences for equally spaced values of t. The second differences found above are approximately constant.

Aquadraticmodelisappropriatebecause$\Delta^2 T$isapproximatelyconstant.

([1])

(c) Newton's forward formula is appropriate because the data start at $t = 0$ with unit spacing. Use the first available first and second differences.

$$T = 8.8 + 4.4t + (-2.2)\frac{t(t-1)}{2!} = 8.8 + 4.4t - 1.1t(t-1) = 8.8 + 5.5t - 1.1t^2.$$

$$T = 8.8 + 5.5t - 1.1t^2.$$

([4])

(d) Substitute the times after September into the model, then compare with the recorded temperatures.

$$T(5) = 8.8 + 5.5(5) - 1.1(5)^2 = 8.8, \qquad T(6) = 8.8 + 5.5(6) - 1.1(6)^2 = 2.2.$$

Itisagoodfitfor October(8.8isclosto8.9), butapoorfitfor November(2.2isfarfro

([2])

(e) The 2022 data have the same first two differences as before, but start at $T = 10.3$ *rather than* 8.8*, so only the constant term changes.*

$$T = 10.3 + 5.5t - 1.1t^2.$$

$$T = 10.3 + 5.5t - 1.1t^2.$$

([1])

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Question 5

Question 5

A bisection spreadsheet has formula

```
=EXP(A2)-A2^2-A2-2
```

in cell B2, and rows ending with $[2.125, -0.2677, 2.25, 0.17524, 2.1875, -0.0598]$. Give the equation, spreadsheet formulae, the next row, a justified root, and the number of additional bisections needed for an interval below 0.0005.

Model solution

(a) The cell formula is the function $f(x)$ evaluated at the left endpoint, so setting it equal to zero gives the required equation.

$$f(x) = e^x - x^2 - x - 2.$$

$$e^x - x^2 - x - 2 = 0.$$

([2])

(b) The midpoint c is the mean of the current endpoints a and b.

$$\text{CellE2} : =(\texttt{A2+C2})/2.$$

([1])

(c) Since $f(a) < 0$ throughout the displayed calculation, if $f(c) > 0$ then c becomes the new right endpoint; otherwise the old right endpoint remains.

$$\text{CellC3} : =\texttt{IF(F2>0,E2,C2)}.$$

([1])

(d) Here $f(2.1875) < 0$ and $f(2.25) > 0$, while the midpoint is 2.21875.

row	a	$f(a)$	b	$f(b)$	c	$f(c)$
6	2.1875	−0.0598	2.25	0.17524	2.21875	0.05423 . . .

$$\text{Row 6 is as shown, with } f(2.21875) \approx 0.0542.$$

([2])

(e) The sign change in row 6 brackets the root. Both endpoints round to the same value to one decimal place, so that is the justified precision.

$$2.1875 < \textit{root} < 2.21875.$$

$$\text{The root is } 2.2 \text{ (1 d.p.)}.$$

([1])

(f) The current interval has length $2.25 - 2.1875 = 0.0625$*. Each bisection halves it, so after* n *more bisections its length is* $0.0625(0.5)^n$*.*

$$0.0625(0.5)^n < 0.0005 \quad \Rightarrow \quad n > \frac{\ln(0.0005/0.0625)}{\ln(0.5)} = 5.96\ldots.$$

6 more bisections are needed.

([3])

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Question 6

Question 6

Values are $f(1.5) = 0.84089$, $f(1) = 1$, $f(2) = 1.18921$. Discuss central and forward differences at $x = 1$. For forward-difference estimates of $f'(2)$ as h halves, the final values are $0.148304, 0.148477$, with difference 0.000173 and ratios tending to 0.5. Determine $f'(2)$ as accurately as possible.

Model solution

(a) A central difference at $x = 1$ needs function values at equal distances on both sides of 1. Both supplied neighbouring x-values lie to its right.

$$\text{It cannot be used : there is no value equally spaced below 1 (for example, } f(0) \text{ is needed with } f(2)).$$

([1])

(b) A forward difference uses $x = 1$ and the next supplied point, $x = 2$.

$$f'(1) \approx \frac{f(2) - f(1)}{2 - 1} = \frac{1.18921 - 1}{1} = 0.18921.$$

$$f'(1) \approx 0.18921.$$

([2])

(c)(i) Successive step lengths are halved. For an error proportional to h^p, successive differences have ratio 2^{-p}. The listed ratios tend to $0.5 = 2^{-1}$.

$$\text{The method has order 1 (first - order convergence).}$$

([2])

(c)(ii) This is expected, since the forward-difference formula is normally first order.

It is not unusual.

([1])

(d) Extrapolate the last estimate to zero step size. If r is the limiting ratio of successive differences, then the remaining error after an estimate X is $D\,r/(1 - r)$, where D is the next difference.

$$f'(2) \approx 0.148477 + 0.000173\frac{0.5}{1 - 0.5} = 0.148650\ldots.$$

The extrapolation removes the leading first-order error, so it is much more accurate than the displayed forward differences.

$$f'(2) \approx 0.14865.$$

([4])

Question 7

Question 7

Approximations to $\int_0^1 f(x)\,dx$ are calculated by midpoint, trapezium and Simpson rules. Given $M_1 = 0.612547$, $T_1 = 1$, $M_2 = 0.639735$, complete the table. Simpson values subsequently end $S_{64} = 0.659015$, with differences $-0.0209, -0.00915, -0.0039, -0.00162, -0.00066$ and ratios tending to 0.4. Give the best integral estimate.

Model solution

(a) The spreadsheet calculations give the missing rule estimates; retain five decimal places as requested.

n	M_n	T_n	S_{2n}
1	0.612547	1	0.74170
2	0.639735	0.80627	0.69525

$$S_2 = 0.74170, T_2 = 0.80627, S_4 = 0.69525.$$

([3])

(b) Simpson estimates $S_2 = 0.74170$ and $S_4 = 0.69525$ agree to one decimal place. No further precision is justified merely from these two estimates.

$$\int_0^1 f(x)\,dx \approx 0.7.$$

([1])

(c) The displayed differences are rounded, but a spreadsheet keeps more digits internally. Its ratio is calculated from those stored values, whereas the calculator calculation uses already-rounded displayed numbers.

Thespreadsheetvalue0.406785isthemorepreciseapproximation.

([2])

(d)(i) The ratios approach a non-zero constant, about 0.4, so the ratio sequence has first-order convergence.

Theratiosconvergewithorder1.

([1])

(d)(ii) Simpson-rule estimates use twice as many subintervals each time. If the method has order p, the ratio of differences tends to 2^{-p}. Since $0.25 < 0.4 < 0.5$,

$$2^{-2} < 0.4 < 2^{-1},$$

theorderofthemethodisbetween1and2.

([2])

(d)(iii) This is unusual: for suitably smooth functions Simpson's rule normally has fourth-order convergence.

It is unusual because Simpson's rule is normally fourth order.

([1])

(e) The ratios suggest $r \approx 0.4$*. Extrapolate the final Simpson value by adding its remaining error, using the final difference* $D = -0.00066$*.*

$$I \approx S_{64} + D\frac{r}{1-r} = 0.659015 - 0.00066\frac{0.4}{0.6} = 0.658575.$$

Using the more precise spreadsheet ratio $r = 0.406785$ *gives* $0.6585624\ldots$ *(and the plausible extrapolated range is about* 0.6585624 *to* 0.65858*). Thus the extrapolation supports the quoted improved estimate.*

$$\int_0^1 f(x)\,dx \approx 0.65856.$$

([4])

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